Stochastic dominance, risk and disappointment: a synthesis

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CES Working Papers

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2014.54RR

Version révisée

Maison des Sciences Économiques, 106-112 boulevard de L'Hôpital, 75647 Paris Cedex 13
http://centredeconomiesorbonne.univ-paris1.fr/
ISSN : 1955-611X

Stochastic dominance, risk and disappointment: a synthesis.

January 2016

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ABSTRACT: In this article, utilities are substituted for monetary values in the definition of second order stochastic dominance (SSD). Doing so yields a family of preorders induced by SSD among which one is the "closest" to the original preorder of preferences. The corresponding utility function is the most likely to be that of the decision maker. It may be defined *before* behavioural axioms are set. Theories of decision making under risk can then be restated in a more general and consistent way. As an example, a new theory of disappointment is developed, which is endowed with three important properties: (a) risk premia are invariant by translation, (b) when constant marginal utility is assumed, preferences are represented by a functional which is the opposite to a convex measure of risk and (c) the functional representing preferences and the utility function can be easily elicited through experimental testing.

RESUME: Une fonction d'utilité est une bijection des loteries "ordinaires" sur des loteries où des "utilités" auront été substituées aux revenus monétaires. On peut définir une dominance stochastique de second ordre sur ces loteries qui est "subjective" parce qu'elle dépend de la fonction retenue. Parmi les préordres induits par ces dominances stochastiques, il en est un qui est le plus "proche" des préférences de l'agent économique considéré. Il correspond à une fonction d'utilité qui apparaît comme étant la plus vraisemblable pour décrire le comportement de l'agent; elle peut être déterminée *avant* que soit mise en place un jeu d'axiomes caractérisant la rationalité des choix en univers risqué. N'importe quelle théorie de la décision peut alors être reformulée compte tenu de ce que cette fonction d'utilité doit être celle de l'agent. A titre d'application, nous reformulons une théorie de la déception où la fonctionnelle représentant les préférences est "loterie-dépendante". Cette théorie possède trois propriétés très intéressantes : (a) les primes de risque sont invariantes par translation (b) la fonctionnelle représentant les préférences n'est autre que l'opposé d'une mesure convexe de rique (Föllmer et Schied 2002), si l'utilité marginale de la richesse est constante et (c) l'on peut déterminer la fonction d'utilité élémentaire à partir de tests empiriques.

JEL classification: D81.
KEY-WORDS: disappointment, risk-aversion, expected utility, risk premium, stochastic dominance, subjective risk.
MOTS-CLES: déception, aversion pour le risque, risque subjectif, prime de risque, utilité espérée.

* This paper is a revised version of the paper entitled "*Risk, stochastic dominance and disappointment: a synthesis*", which was presented at the 2014 FUR conference in Rotterdam (1st July) and at the 2015 SAET conference in Cambridge (UK, July 28th).

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1 *Introduction*

According to expected utility theory –henceforth EU theory–, a decision maker maximizes the expected utility of his wealth. He is risk-averse (prone) if his utility function is concave (convex). Risky outcomes then exhibit positive (negative) premia. Despite its appealingness, this result remains questionable since it may be argued that, whatever his attitude towards risk, the EU decision maker actually behaves in the same way. Indeed, in all cases, he takes into account nothing but the average value of random welfares. In particular, he will be indifferent between a certain welfare $\mathbf{U}$ and a risky welfare U whose expected value $\mathbf{E}[U]$ is equal to $\mathbf{U}$. Finally, when utils are taken into account instead of monetary values, the EU decision maker seems to behave as if he were indifferent between the two branches of an alternative, one of which is clearly less risky than the other.

A way of making this paradox vanish is to consider that risk aversion (proneness) exists if and only if –henceforth *iff*– the risk premium $\mathbf{E}[U] - \mathbf{U}$ is positive (negative). This will be the case if preferences are defined over random utilities, what will be done in this article. It will make sense because the utility function will have been defined *before* the axioms characterizing the behaviour of the decision maker are set. The utility function will be derived from comparing the preorder of preferences $\succsim$ to the preorders induced by *subjective second order stochastic dominance* –henceforth SSSD–.

SSSD is just the same as the usual second order stochastic dominance –henceforth SSD– except that the utility of an outcome is then substituted for its monetary value. Of course, the definition of SSSD depends on the considered utility function $u(.)$. It will be labelled $\succcurlyeq_2^u$. Like SSD, it is a partial preorder.

In principle, violations of SSSD must be ruled out when the decision maker is rational and risk-averse. Hence we shall focus on preorders induced by SSSD which never disagree with $\succsim$. It will be shown that, among the utility functions characterizing these preorders, there exists one function such that the size of the subset of pairs of lotteries for which $\succsim$ and $\succcurlyeq_2^u$ are in accordance is the largest. This utility function is clearly the most likely to be that of the decision maker.

Once the utility function has been defined, the decision maker may be viewed as making a risky choice among random utilities, *i.e.* among random variables whose consequences are valued in utils. Behavioural axioms may then be set.

As an example, we develop an original theory of disappointment. The reason for this is both psychological and economic.

It is often emphasized, in *psychological literature*, that (a) disappointment (elation) is experimented, once a decision has been taken, when the chosen option turns out to be worse (better) than expected (see *e.g.* Mellers 2000), (b) that it is the most frequently experimented emotion (see Weiner *et alii* 1979) and (c) that disappointment is the most powerful among the negative emotions which are experimented (see Schimmack and Diener 1997).[1] Moreover, as Frijda

[1] There is a lot of empirical evidence which supports this view (see *e.g.* Van Dijk and Van der Pligt 1996, Zeelenberg *et alii* 2002, Van Dijk *et alii* 2003).

(1994) points out, “actual emotion, affective response, anticipation of future emotion can be regarded as the primary source of decisions”. To sum up, it is most likely that expected elation/disappointment plays an important role in decision making.

This role was first formalized independently by Bell (1985) and Loomes and Sugden (1986). Despite its earliness, their approach has revealed surprisingly close to the analyses mentioned above. However, their disappointment models have been somewhat neglected in the *economic literature*, probably because they lack an axiomatic framework.

In the meantime, other disappointment models have been developed. Unfortunately, most of them also lack an axiomatic basis. A major exception is Gul (1991) who developed an implicit expected utility model of disappointment where the certainty equivalent of the lottery plays the role of reference level. However, Gul’s theory does not guarantee that the utility function which is derived from his axiomatic is the most likely to be that of the decision maker.

In contrast, our theory of disappointment has been developed in order to meet this property. It will be called LS-theory since it generates a set of models –called LS-models– which are close to that of Loomes and Sugden (1986) although the utility function of the decision-maker is now lottery-dependent.[2] LS-theory also meets four other important properties: (*a*) risk premia are invariant by translation;[3] (*b*) in particular, when constant marginal utility is assumed, preferences are represented by a functional which is nothing but the opposite to a convex measure of risk (Föllmer and Schied 2002); (*c*) quantities of risk are identified to centered moments of the distribution of the utility of the decision maker’s wealth. The global risk premium is then a sum of elementary risk premia, each of which is the product of a quantity of risk by a specific coefficient of risk aversion. When the functional is quadratic, the risk premium comes down to the product of the variance of the utility of the decision maker’s wealth by risk aversion; (*d*) the functional representing preferences and the utility function can be easily elicited through experimental testing.

The rest of this article is organized as follows: in Section 2, new concepts (subjective stochastic dominances, subjective risk,...) will be introduced and it will be shown how a utility function can be derived from preferences over random utilities. In the next sections, a theory of disappointment is developed: Section 3 is devoted to the study of the particular case of constant marginal utility. In Section 4, the general theory is presented. Section 5 concludes.

[2] Since the independence property is met only for lotteries exhibiting the same expected utility.

[3] Unless constant marginal utility is assumed, risk premia are valued in utils and the invariance property matches this definition.

2 *How to derive a utility function from preferences*

2.1 Definitions

In this article, the decision maker is assumed to face a problem of risky choice over a set $\mathfrak{X} = \{X, Y, Z, ...\}$ of random variables mapping a set of "states of nature" Ω on to a set of outcomes which is identified to an interval $[a, b]$ of $\mathbb{R}$. The outcomes are identified to monetary prizes. An element of $\mathfrak{X}$ will be called a lottery. Any lottery $X \in \mathfrak{X}$ is endowed with a probability distribution. The subset of these probability distributions will be labelled $\mathfrak{L}$. A probability distribution will be identified to its cumulative distribution function –henceforth *c.d.f.*–. The *c.d.f.* of X is labelled $F_X(x)$ and its expected value $\mathbf{E}[X]$. The random variable whose outcome is x with certainty, will be labelled δ_x.

When the set of events is finite, a random variable $X \in \mathfrak{X}$ has a finite support $\{x_1, x_2, ..., x_N\}$ where $x_1 < x_2 < ... < x_N$; it will be called a *simple lottery* and labelled: $X = [x_1, x_2, ..., x_N; p_1, p_2; ..., p_N]$ where $p_n = \Pr(X = x_n) \geq 0$ and $\sum_{n=1}^{N} p_n = 1$.

A decision-maker has a preference relation over $\mathfrak{X}$. His preferences will be denoted $\succsim$ (weak preference). The acronym $\succ$ ($\sim$) will be used for strong preference (indifference). The certainty equivalent of $X \in \mathfrak{X}$ is the certain outcome which is indifferent to X. It is labelled $\mathbf{c}(X)$ (*i.e.* $X \sim \delta_{\mathbf{c}(X)}$).

A *utility function* will be defined as a *derivable increasing function* mapping $[a, b]$ on to $[0, 1]$.[4] Actually, utility functions will also be assumed to be *convex or concave* over $[a, b]$. However, this assumption will be relaxed later (See Appendix 1). The subset of utility functions will be labelled $\mathbb{U}$.

A utility function $u(.)$ may also be viewed as defining a one-to-one correspondance mapping $\mathfrak{X}$ on to $\mathfrak{U}$ where $\mathfrak{U} = \{U, V, ...\}$ is the set of random variables mapping Ω on to $[0, 1]$. An element of $\mathfrak{U}$ is defined by: $U = u(X)$; it will be called a **u**-lottery. The *c.d.f.* of $U \in \mathfrak{U}$ will be labelled $G_U(.)$, its certainty equivalent $\mathfrak{c}(U)$ and its expected value $\mathbb{E}[U] = \int_0^1 z dG_U(z)$.

Finally, recall that stochastic dominances are partial pre-ordering relations over $\mathfrak{X}$, which are defined as follows:

$X \succcurlyeq_n Y \Longleftrightarrow F_X^n(x) \leq F_Y^n(x)$ ***and*** $F_X^k(b) \leq F_Y^k(b)$, $\forall x \in [a, b]$ $\forall k \leq n - 1$.

where $X \succcurlyeq_n Y$ stands for "X dominates Y by n-th order stochastic dominance" and where

$F_X^1(x) \overset{def}{=} F_X(x)$; $F_X^n(x) \overset{def}{=} \int_a^x F_X^{n-1}(t)dt$,

The acronym $\succ_n$ will be substituted for $\succcurlyeq_n$ when the dominance is strict.

In particular, the partial preorder induced by first-order (second order) stochastic dominance (henceforth FSD (SSD)) are characterized by the following equivalences

$X \succcurlyeq_1 Y \Longleftrightarrow F_X(x) \leq F_Y(x) \quad \forall x \in [a, b]$.

$X \succcurlyeq_2 Y \Longleftrightarrow \int_a^x (F_X(t) - F_Y(t))\, dt \leq 0, \ \forall x \in [a, b]$.

[4] As a consequence, one may say that it is normalized. The reason for such a choice will become apparent later.

A particular case of stochastic dominance is an increase in risk (see Ekern 1980) which is defined as follows:

$X \succcurlyeq_n^R Y \Longleftrightarrow F_X^n(x) \leq F_Y^n(x)$ ***and*** $F_X^k(b) = F_Y^k(b), \forall\, x \in [a,b]\ \forall\, k \leq n-1.$
where $X \succcurlyeq_n^R Y$ stands for "Y is a nth-degree increase in risk over X". For example, a 2nd-degree increase in risk coincides with a mean preserving spread, (see Rothschild and Stiglitz 1970).

2.2 Subjective stochastic dominances and subjective risk

Some new concepts are now going to be defined: that of subjective stochastic dominance and that of subjective risk.

Definition 1 (subjective stochastic dominance). *Let $u(.)$ be a utility function, let $X_i \in \mathfrak{X}$ and let $U_i = u(X_i)$ for $i = 1, 2$. It is equivalent to state that:*

(a) X_1 dominates X_2 by SFSD (SSSD) or that

(b) U_1 dominates U_2 by FSD (SSD).

Definition 2. (subjective risk) *Let $u(.)$ be a utility function, let $X_i \in \mathfrak{X}$ and let $U_i = u(X_i)$ for $i = 1, 2$. It is equivalent to state that:*

(a) X_2 is a second degree increase in subjective risk over X_1 or that

(b) U_2 is a second degree increase in risk over U_1, (i.e. U_2 is a mean preserving spread of U_1).

From now on, SFSD (SSSD) will be labelled $\succcurlyeq_1^u$ ($\succcurlyeq_2^u$).The acronym $\succ_n^u$ will be substituted for $\succcurlyeq_n^u$ when the dominance is strict (for $n = 1, 2$). As already indicated in the introductory section, the SSSD property is just the same as the usual SSD property except that the utility of any outcome, namely $u(x)$, is substituted for its monetary value, namely x, in the corresponding tests. SSSD may be characterized as follows:

Proposition 1 (characterization of SSSD). *Let $(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X}$. Let $u(.)$ be a utility function and let $U_i = u(X_i)$ for $i = 1, 2$. It is equivalent to state:*

(a) $X_1 \succcurlyeq_2^u X_2$ or

(b) $\int_a^z u'(x)\,(F_{X_1}(x) - F_{X_2}(x))dx \leq 0$ for any $z \in [a,b]$.

Proof. It is given in Appendix 4.□

Clearly, the definitions of subjective stochastic dominance and of subjective risk depend on the considered utility function $u(.)$. Hence, the SSD and the SSSD dominance properties are not necessarily met simultaneously.[5] As a consequence, one can define $\mathbb{X}_{\mathbf{2}}^{u+}$ ($\mathbb{X}_{\mathbf{2}}^{u-}$) as the subset of pairs of lotteries (X_1, X_2) over which the two preorders, $\succcurlyeq_2^u$ and $\succsim$ coincide (disagree), what formally reads:

$\mathbb{X}_{\mathbf{2}}^{u+} = \{(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X} \mid X_1 \succcurlyeq_2^u X_2 \text{ and } X_1 \succsim X_2\}$ and
$\mathbb{X}_{\mathbf{2}}^{u-} = \{(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X} \mid X_1 \succcurlyeq_2^u X_2 \text{ and } X_2 \succ X_1\}$.

[5] However, FSD dominance is a property which is conservative through the change of random variable: $Y = u(X)$ since, by assumption, $u'(.)$ is positive.

2.3 Closeness to the preorder of preferences

Intuitively, the preorder $\succcurlyeq_2^u$ will be all the more "close" to the total preorder $\succsim$ that the size of $\mathbb{X}_{\mathbf{2}}^{u-}$ is smaller and that the size of $\mathbb{X}_{\mathbf{2}}^{u+}$ is larger *i.e.* that violations of SSSD are more scarce and that the two preorders more often agree. Consequently, we set the following definition:

Definition 3. *The preorder $\succcurlyeq_2^u$ is closer to the total preorder $\succsim$ than the preorder $\succcurlyeq_2^v$ iff either:*
(a) $\mathbb{X}_{\mathbf{2}}^{u-} \subset \mathbb{X}_{\mathbf{2}}^{v-}$ or:
(b) $\mathbb{X}_{\mathbf{2}}^{u-} = \mathbb{X}_{\mathbf{2}}^{v-}$ and $\mathbb{X}_{\mathbf{2}}^{v+} \subseteq \mathbb{X}_{\mathbf{2}}^{u+}$.
The acronym "$\succcurlyeq_2^u \mathcal{C}l \succcurlyeq_2^v$" will stand for "$\succcurlyeq_2^u$ is closer to $\succsim$ than $\succcurlyeq_2^v$".

It is obviously a partial preorder. The scarcity of SSSD violations is clearly favoured, in the above definition, at the expense of the frequency of the concordances of the two preorders. An important result then consists in the following proposition:

Proposition 2. *Let $u(.)$ and $v(.)$ be two utility functions: if $u(.)$ is more concave (i.e. less convex) than $v(.)$, then:*

$$X_1 \succcurlyeq_2^v X_2 \Rightarrow X_1 \succcurlyeq_2^u X_2 \tag{1}$$

or, equivalently:

$$\mathbb{X}_{\mathbf{2}}^{v+} \subseteq \mathbb{X}_{\mathbf{2}}^{u+} \text{ and } \mathbb{X}_{\mathbf{2}}^{v-} \subseteq \mathbb{X}_{\mathbf{2}}^{u-}$$

Proof. It is given in Appendix 4.□

As indicated in the above proposition, making a utility function more concave increases the size of $\mathbb{X}_{\mathbf{2}}^{u+}$ and, simultaneously, that of $\mathbb{X}_{\mathbf{2}}^{u-}$. In other words, increasing the size of $\mathbb{X}_{\mathbf{2}}^{u+}$ will often be obtained at the expense of increasing the size of $\mathbb{X}_{\mathbf{2}}^{u-}$ –for instance making $\mathbb{X}_{\mathbf{2}}^{u-}$ become non-empty–.

As a consequence, making a utility function more concave will be desirable only when the function is convex enough. Symmetrically, when the utility function is markedly concave, violations of SSSD will increase if the function is made still more concave.

Finally, making utility functions become more and more concave will make their associate preorders $\succcurlyeq_2^u$ become closer and closer to $\succsim$ as far as the utility functions remain convex enough. This will be the case if they remain *consistent*, in the sense given below:

Definition 4 (consistency). *A utility function is consistent iff the preorder $\succcurlyeq_2^u$ never disagrees with the preorder of preferences, what formally reads: $\mathbb{X}_{\mathbf{2}}^{u-} = \varnothing$. The subset of consistent (inconsistent) utility functions will be labelled $\mathbb{U}_{\mathrm{C}}$ ($\mathbb{U}_{\mathrm{I}}$).*

Definition 4 clearly implies that the subset of consistent utility functions is such that *a preorder associated with a consistent function is closer to $\succsim$ than any preorder associated with an inconsistent function*, what formally reads:

$$u(.) \in \mathbb{U}_{\mathrm{C}} \text{ and } v(.) \in \mathbb{U}_{\mathrm{I}} \Longrightarrow \succcurlyeq_2^u \mathcal{C}l \succcurlyeq_2^v$$

We may then focus on consistent utility functions and look for a consistent utility function $\mathbf{u}(.)$, such that $\mathbb{X}_{\mathbf{2}}^{\mathbf{u}+}$ is as large as possible. As a preliminary, note that Proposition 2 and Definition 4 imply the following corollary:

Corollary 1. *Let $u(.)$ and $v(.)$ be two consistent utility functions such that $u(.)$ is more concave (i.e. less convex) than $v(.)$, then the following relations are valid:*

$$X_1 \succcurlyeq_2^v X_2 \Longrightarrow X_1 \succcurlyeq_2^u X_2 \tag{2}$$

$$\mathbb{X}_{\mathbf{2}}^{v+} \subseteq \mathbb{X}_{\mathbf{2}}^{u+} \text{ and } \mathbb{X}_{\mathbf{2}}^{v-} = \mathbb{X}_{\mathbf{2}}^{u-} = \varnothing$$

Proof. Corollary 1 is a direct consequence of Proposition 2 and of Definition 4.□

The above result is going to be used to define a new concept: that of likely utility function.

2.4 Likely utility functions.

Actually, we get the following result which applies to *consistent* utility functions.

Proposition 3. (likely utility function). *There exists a unique consistent utility function $\mathbf{u}(.)$ such that the preorder $\succcurlyeq_2^{\mathbf{u}}$ is the closest to the preorder of preferences $\succsim$ among the preorders induced by consistent utility functions, what formally reads:*

$$\forall u(.) \in \mathbb{U}_{\mathrm{C}}, \quad \succcurlyeq_2^{\mathbf{u}} \ \mathcal{C}l \ \succcurlyeq_2^u \tag{3}$$

where $\mathbb{U}_{\mathrm{C}}$ is the subset of consistent utility functions. It is the superior enveloppe of the consistent concave or convex utility functions. It will be called the likely utility function of the decision maker.

Proof. The proof is given in Appendix 4.□.

If violations of (standard) SSD occur,[6] any concave utility function must be inconsistent, otherwise Proposition 2 would not hold. In contrast, violations of SSSD will vanish for a convex enough function and the likely utility function is convex.

If violations of (standard) SSD do not occur, then concave consistent utility functions do exist and the likely utility function is concave. A geometrical illustration of the above discussion is given on Figures 1A and 1B.

Finally a likely utility function is concave or convex, what may be viewed as a too strong property. However, Proposition 3 may be slightly modified to allow for *piecewise concave or convex likely utility functions*. To spare space, this generalization is postponed until Appendix 1.

Anyway, according to Proposition 3, the likely utility function is such that $\mathbb{X}_{\mathbf{2}}^{\mathbf{u}+}$ is as large as possible. It is the most likely to be the actual utility function of the decision maker since it provides the "best" consistency with the preorder of preferences. This result is all the more interesting that, as will be shown in Appendix 3, the likely utility function of a decision maker can be *elicited* from binary choices over simple lotteries.

[6] Recall that although violations of SSSD are ruled out for the preorders associated with consistent utility functions, violations of (standard) SSD may occur even though the subset of consistent utility functions is not empty.

A last remark is that assuming that utility functions are normalized, what has been made at the beginning of this article, is not really restrictive. Indeed, if $u(.)$ is as concave (or, equivalently, as convex) as $v(.)$, or, equivalently, if $u(.)$ is an affine and increasing function of $v(.)$, then $\succcurlyeq_2^u$ is as close to $\succsim$ as $\succcurlyeq_2^v$. Indeed, we are interested in nothing but the equivalence classes of the binary relation "$u(.)$ is as concave (or, equivalently, as convex) as $v(.)$". A set of class representatives will consist in a subset of utility functions meeting the same two normalization conditions, which have been particularized as $u(a) = 0$ and $u(b) = 1$.

Finally, from the analysis developed above, we get that a rational investor is endowed with a likely utility function. It will be labelled $\mathbf{u}(.)$.

2.5 A generalized EU theory

A simple way of generalizing EU theory consists in substituting utils for monetary values in its axiomatic. Preferences over **u**-lotteries (labelled $\succsim_{\mathrm{u}}$) are then canonically defined by the following equivalence:

$X_1 \succsim X_2 \Longleftrightarrow U_1 \succsim_{\mathrm{u}} U_2$

where $U_i = u(X_i)$ for $i = 1, 2$. The axioms of EU theory may then be modified as indicated below:

Axiom I (ordering of $\succsim_{\mathrm{u}}$). *The binary relation $\succsim_{\mathrm{u}}$ is a total preorder over the set $\mathfrak{U}$ of* **u***-lotteries.*

Axiom II (continuity of $\succsim_{\mathrm{u}}$). *For any* **u***-lottery the sets $\{Z \in \mathfrak{U} \mid Z \succsim_{\mathrm{u}} U\}$ and $\{Z \in \mathfrak{U} \mid U \succsim_{\mathrm{u}} Z\}$ are closed in the topology of weak convergence.*

Axiom III (independence axiom). *The following implications hold:*

$\forall U_1, U_2, U_3 \in \mathfrak{U}, \forall \lambda \in [0,1], U_2 \succ_{\mathrm{u}} U_3 \Longrightarrow \lambda U_1 \oplus (1-\lambda) U_2 \succ_{\mathrm{u}} \lambda U_1 + (1-\lambda) U_3$

$\forall U_1, U_2, U_3 \in \mathfrak{U}, \forall \lambda \in [0,1], U_2 \sim_{\mathrm{u}} U_3 \Longrightarrow \lambda U_1 \oplus (1-\lambda) U_2 \sim_{\mathrm{u}} \lambda U_1 + (1-\lambda) U_3$

As a consequence, we get the well-known following representation theorem:

Proposition 4. (expected utility representation theorem for $\succsim_{\mathrm{u}}$ over $\mathfrak{U}$). *Under Axioms I to III, the preorder of preferences of a rational decision-maker –labelled $\succsim_{\mathrm{u}}$– can be represented over $\mathfrak{U}$ by the following lottery-dependent functional*:

$$\mathcal{V}(U) \overset{def}{=} \int_0^1 v(z)\, dG_U(z) = \mathbb{E}[v(U)] \tag{4}$$

where $v(.)$ is a continuous and increasing function which is defined up to an increasing affine function.

Proof. See Fishburn (1970) □

From now on, the two normalization conditions $v(0) = 0$ and $v(1) = 1$ will be assumed to be met, and, consequently, function $v(.)$ will be unique and well defined. Finally, note that (4) may be rewritten as:

$$\mathcal{U}(X) \overset{def}{=} \int_a^b v(\mathbf{u}(x))\, dF_X(x) = \mathbf{E}[v(\mathbf{u}(X))] \tag{5}$$

The function $v \circ \mathbf{u}(.)$ is clearly continuous and increasing. It may be concave, convex or neither concave nor convex[7] and it is interesting to point out that, because of its flexibility, the functional (5) is compatible with some of the well-kown anomalies of financial theory. However, this issue is beyond the scope of this article.

Unfortunately, the above generalized EU theory is not endowed with all the suitable properties of a theory of decision making under risk: in particular, it does not allow for risk premia invariance by translation.[8] As a consequence, we have developped an alternative theory which is nothing but a new theory of disappointment.

3 *Disappointment with constant marginal utility*

To make things clearer, we first focus on a particular case: that of constant marginal utility. Because of the normalization conditions, the likely utility function will be defined as: $\mathbf{u}(x) = (x - a)/(b - a)$. A decision maker will now care for monetary outcomes.

3.1 The axiomatic of a simplified theory of disappointment

Preferences may then be indifferently defined over $\mathfrak{X}$ or over $\mathfrak{U}$ since the two sets may now be identified.[9] As a consequence, the new theory includes the two first axioms of EU theory. They may be stated as indicated below.

Axiom 1. (ordering of $\succsim$) *The binary relation $\succsim$ is a total preorder over $\mathfrak{X}$.*

Axiom 2. (continuity of $\succsim$) *For any lottery $X \in \mathfrak{X}$ the sets $\{Z \in \mathfrak{X} \mid Z \succsim X\}$ and $\{Z \in \mathfrak{X} \mid X \succsim Z\}$ are closed in the topology of weak convergence.*

As usual, elation (disappointment) will occur when the realized outcome x is higher (lower) than a reference level $\overline{\mathbf{x}}$. The reference level may be viewed as a "prior expectation" of the value of the lottery, which is likely to be an average of *ex-post* outcomes, for instance their expected value $\mathbf{E}[X]$. Elation or disappointment will then depend on the spread $x - \mathbf{E}[X]$.

Equivalently, one can say that the certainty equivalent of a lottery will be its expected value minus a risk premium which is the expectation of a function of the spread, $\mathbf{E}[\mathcal{E}(X - \mathbf{E}[X])]$. The premium will be negative (positive) if the decision maker is risk-averse (prone). As a consequence, the certainty equivalent of the $(\lambda, 1-\lambda)$-mixing of two lotteries exhibiting the same expected value $\overline{\mathbf{x}} = \mathbf{E}[X]$, will be the corresponding convex combination of the certainty equivalents of the two considered lotteries. Hence, the following axiom will be set.

[7] In the last case, it may be piecewise concave/convex.

[8] This drawback is shared by almost all theories of decision making under risk.

[9] Since they are isomorphic through the change of variable: $U = \mathbf{u}(X)$.

Axiom 3. (linearity of certainty equivalents of lotteries belonging to $\mathfrak{X}_{\overline{\mathbf{x}}}$) *The certainty equivalent of the $(\lambda, 1-\lambda)$-mixing of two lotteries exhibiting the same expected value is the $(\lambda, 1-\lambda)$-convex combination of their certainty equivalents, what formally reads:*

$$\forall X, Y \in \mathfrak{X}_{\overline{\mathbf{x}}},\ \forall \lambda \in [0,1]\,,\ \mathbf{c}\left(\lambda X \oplus (1-\lambda)\,Y\right) = \lambda \mathbf{c}\left(X\right) + (1-\lambda)\,\mathbf{c}\left(Y\right)$$

where $\mathfrak{X}_{\overline{\mathbf{x}}} = \{X \in \mathfrak{X} \,|\mathbf{E}\left[X\right] = \overline{\mathbf{x}}\}$.

The axiom clearly implies that *the independence property is met over any subset of lotteries exhibiting the same expected value.* However, the independence property is weaker than the above linearity property since the degenerate lottery $\delta\left(\lambda X \oplus (1-\lambda)\,Y\right)$ does not belong to $\mathfrak{X}_{\overline{\mathbf{x}}}$. Anyway, the above axioms imply the following result:

Proposition 5. *Under Axioms 1 to 3, the total preorder of preferences of a decision maker $\succsim$ may be represented over $\mathfrak{X}_{\overline{\mathbf{x}}}$ by a continuous real-valued function $\mathcal{U}_{\overline{\mathbf{x}}}(.)$ which is linear, i.e.:*

$$\forall X, Y \in \mathfrak{X}_{\overline{\mathbf{x}}},\ \forall \lambda \in [0,1]\,,\ \mathcal{U}_{\overline{\mathbf{x}}}\left(\lambda X \oplus (1-\lambda)\,Y\right) = \lambda \mathcal{U}_{\overline{\mathbf{x}}}\left(X\right) + (1-\lambda)\,\mathcal{U}_{\overline{\mathbf{x}}}\left(Y\right) \quad (6)$$

Furthermore $\mathcal{U}_{\overline{\mathbf{x}}}(.)$ is defined up to an affine and positive transformation.

Proof. Since the subset $\mathfrak{X}_{\overline{\mathbf{x}}}$ is a mixture set, the same proof as the one given in Fishburn (1970) may be used.□

Actually, as shown in the next proposition, a stronger result is available.

Proposition 6 (expected utility representation theorem for $\succeq$ over $\mathfrak{X}_{\overline{\mathbf{x}}}$). *Under Axioms 1 to 3, the functional $\mathcal{U}_{\overline{\mathbf{x}}}(.)$ may be defined as:*

$$\mathcal{U}_{\overline{\mathbf{x}}}\left(X\right) \overset{def}{=} \int_a^b u_{\overline{\mathbf{x}}}(x) dF_X(x) \quad (7)$$

where $u_{\overline{\mathbf{x}}}(.)$ is a continuous and increasing function mapping $[a,b]$ on to $[u_{\overline{\mathbf{x}}}(a), u_{\overline{\mathbf{x}}}(b)]$ which is defined up to an affine and positive transformation.

Proof. If only simple lotteries were considered, the three above axioms would clearly be sufficient for the above proposition to hold. Actually, the proposition holds even when the whole set $\mathfrak{X}_{\overline{\mathbf{x}}}$ is taken into account. The proof is given in Appendix 4. Note that neither a dominance axiom nor a monotonicity axiom need then to be set.□

We now set the following normalization conditions:

$$u_{\overline{\mathbf{x}}}(\overline{\mathbf{x}}) = \overline{\mathbf{x}} \ \ \textit{and} \ \ \pi\left(\overline{\mathbf{x}}\right) u_{\overline{\mathbf{x}}}(b) + \left(1 - \pi\left(\overline{\mathbf{x}}\right)\right) u_{\overline{\mathbf{x}}}(a) = \mathbf{c}(X_{\overline{\mathbf{x}}}) \quad (8)$$

where:

$$X_{\overline{\mathbf{x}}} \overset{def}{=} \left[a, 1-\pi\left(\overline{\mathbf{x}}\right); b, \pi\left(\overline{\mathbf{x}}\right)\right] \ ; \ \pi\left(\overline{\mathbf{x}}\right) \overset{def}{=} \left(\overline{\mathbf{x}} - a\right) / \left(b - a\right).$$

As a consequence, $u_{\overline{\mathbf{x}}}(.)$ is, from now on, unique and well defined. However, the above results do not provide a method for comparing two lotteries whose expected values differ because it is not guaranteed that the values of the functional representing the preferences over $\mathfrak{X}_{\overline{\mathbf{x}}}$ (*i.e.* $\mathcal{U}_{\overline{\mathbf{x}}}(.)$) are compatible with those of the functional representing the preferences over $\mathfrak{X}_{\overline{\mathbf{y}}}$ (*i.e.* $\mathcal{U}_{\overline{\mathbf{y}}}(.)$). However, since

Axiom 3 is stronger than what would have been a weakened independence axiom yielding only the independence property over $\mathfrak{X}_{\overline{\mathbf{x}}}$, this does occur.

Indeed, let $X \in X_{\overline{\mathbf{x}}}$ and λ be defined by: $X \sim \lambda\delta_{\overline{\mathbf{x}}} \oplus (1-\lambda)\, X_{\overline{\mathbf{x}}}$. Then $\delta_{\overline{\mathbf{x}}}$ (X) clearly SSD dominates X $(X_{\overline{\mathbf{x}}})$. As a consequence, we have $\delta_{\overline{\mathbf{x}}} \succ X \succ X_{\overline{\mathbf{x}}}$ and λ belongs to $[0,1]$.

Finally, the certainty equivalent of X is equal to :

$\mathbf{c}\,(X) = \mathbf{c}\,(\lambda\delta_{\overline{\mathbf{x}}} \oplus (1-\lambda)X_{\overline{\mathbf{x}}})$.

Next, Axiom 3 implies that: $\mathbf{c}\,(X) = \lambda\mathbf{c}\,(\delta_{\overline{\mathbf{x}}}) + (1-\lambda)\,\mathbf{c}\,(X_{\overline{\mathbf{x}}})$,
and, thanks to the normalization conditions (8) that:

$\mathbf{c}\,(X) = \lambda\mathcal{U}_{\overline{\mathbf{x}}}\,(\delta_{\overline{\mathbf{x}}}) + (1-\lambda)\,\mathcal{U}_{\overline{\mathbf{x}}}\,(X_{\overline{\mathbf{x}}})$.

Moreover, Proposition 4 implies that:

$\mathcal{U}_{\overline{\mathbf{x}}}\,(\lambda\delta_{\overline{\mathbf{x}}} \oplus (1-\lambda X_{\overline{\mathbf{x}}})) = \lambda\mathcal{U}_{\overline{\mathbf{x}}}\,(\delta_{\overline{\mathbf{x}}}) + (1-\lambda)\,\mathcal{U}_{\overline{\mathbf{x}}}\,(X_{\overline{\mathbf{x}}})$
and, consequently: $\mathcal{U}_{\overline{\mathbf{x}}}\,(X) = \mathbf{c}\,(X)$
or, equivalently:

$$\mathcal{U}_{\mathbf{E}[X]}\,(X) = \mathbf{c}\,(X) \tag{9}$$

Finally, we are well able to compare lotteries whose expected values differ as indicated in the following proposition:

Proposition 7. (lottery-dependent expected utility representation theorem for $\succsim$ over $\mathfrak{X}$) *Under Axioms 1 to 3, the preorder of preferences $\succsim$ may be represented over $\mathfrak{X}$ by the following lottery-dependent functional*:

$$\mathcal{U}\,(X) \overset{def}{=} \mathcal{U}_{\mathbf{E}[X]}\,(X) = \int_a^b u_{\mathbf{E}[X]}(x)dF_X(x) \tag{10}$$

where $u_{\mathbf{E}[X]}(x)$ is continuous and increasing with respect to x and meets the normalizing conditions (8)

Proof. Eq (9) implies that $\mathcal{U}(.)$ well represents the preorder of preferences $\succsim$ over $\mathfrak{X}$. $\square$

Unfortunately, the above functional remains far too general. Hence, in the next subsection we particularize $u_{\mathbf{E}[X]}(.)$ to provide a more operational specification. This will be done through imposing an additional condition to preferences: the invariance of risk premia by translation.

3.2 Invariance of risk premia by translation

The risk premium of an arbitrary lottery $X \in \mathfrak{X}$ then reads:

$$\mathbf{RP}(X) \overset{def}{=} \mathbf{E}\,[X] - \mathbf{c}\,(X) = \mathbf{E}\,[X] - \int_a^b u_{\mathbf{E}[X]}(x)dF_X(x)$$

It is commonly assumed, in the literature dealing with banking regulation, that risk premia should be invariant by translation, *i.e.*, that

$$\mathbf{RP}(X + \Delta x) = \mathbf{RP}(X)$$

This invariance property is guaranteed by the following necessary and sufficient condition.

Proposition 8. *The risk premium of an arbitrary lottery $X \in \mathfrak{X}$ is endowed with the property of invariance by translation iff:*

$$u_{\overline{\mathbf{x}}}(x) = \overline{\mathbf{x}} + \mathcal{E}\left(x - \overline{\mathbf{x}}\right)$$

where $\mathcal{E}(.)$ is continuous and increasing and meets the following requirements:

$$\mathcal{E}\left(0\right) = 0 \ ; \ \frac{\overline{\mathbf{x}} - a}{b - a}\left(b + \mathcal{E}\left(b - \overline{\mathbf{x}}\right)\right) + \frac{b - \overline{\mathbf{x}}}{b - a}\left(a + \mathcal{E}\left(a - \overline{\mathbf{x}}\right)\right) = \mathbf{c}\left(X_{\overline{\mathbf{x}}}\right) \tag{11}$$

Proof. It is given in Appendix 4.□

From now on, an *elation* function will be defined as a continuous and increasing function $\mathcal{E}(.)$ which meets the above requirements. Its opposite $\mathcal{D}(.) = -\mathcal{E}(.)$ will be called a *disappointment* function. Finally, we shall set the following definition:

Definition 5 (rational decision maker). *A decision maker is rational if he obeys Axioms 1 to 3 and if risk premia are invariant by translation.*

As a consequence, the following corollary is valid.

Corollary 2. *The preorder of the preferences of a rational decision maker are represented over $\mathfrak{X}$ by the following lottery-dependent functional:*

$$\mathcal{U}(X) = \mathbf{E}[X] - \int_a^b \mathcal{D}(x - \mathbf{E}[X])dF_X(x) \tag{12}$$

where $\mathcal{D}(.)$ is a disappointment function.

Proof. It is a direct consequence of Propositions 7 and 8.□

Conversely, if $\mathcal{D}(.)$ is a disappointment function, the corresponding functional $\mathcal{U}(.)$ may be viewed as representing the preferences of a rational decision maker.

3.3 Risk-averse decision makers

Now consider *risk-averse* decision makers. According to Rothschild and Stiglitz (1970), a decision-maker is *risk-averse* if he prefers X to any mean preserving spread of X.[10] In particular, any non-degenerate lottery $X \in \mathfrak{X}$ will exhibit a positive risk premium if X is not degenerate, giving rise to the following equivalence:

$$X \text{ non degenerate} \Longleftrightarrow \int_a^b \mathcal{D}(x - \mathbf{E}[X])dF_X(x) = \mathbf{E}\left[\mathcal{D}\left(X - \mathbf{E}\left[X\right]\right)\right] > 0 \tag{13}$$

and the consistency of risk and risk aversion is guaranteed by the next proposition:

[10] Risk aversion in the sense of Rothschild and Stiglitz is more often denominated *strong risk aversion* whereas *weak risk aversion* characterizes the behaviour of a decision-maker who always prefers $\delta_{\mathbf{E}[X]}$ to X. Strong risk aversion clearly implies weak risk aversion which, in its turn, implies that risk premia are positive.

Proposition 9. *A rational decision maker is risk-averse iff his preferences are represented by a functional $\mathcal{U}(.)$ defined by (12) where $\mathcal{D}(.)$ is a strictly convex disappointment function.*

Proof. If $\mathcal{D}(.)$ is strictly *convex*, then (13) will hold. This is a direct consequence of Jensen's inequality. Conversely, if (13) holds for *all* $X \in \mathfrak{X}$, then $\mathcal{D}(.)$ must be strictly *convex*.[11] $\square$

Two other definitions of the consistency of risk and risk aversion may be considered. According to the first one, a risk premium should depend positively on a quantity of risk and on risk aversion. For instance, a risk premium is often defined as the product of risk by risk aversion.

In the new theory, if the disappointment function is infinitely derivable, the risk premium of X may be rewritten as:

$$\mathbf{RP}\left(X\right) = \textstyle\sum_{p=2}^{+\infty} \mathbf{E}\left[\left(X - \mathbf{E}\left[X\right]\right)^{p}\right] \mathcal{D}^{(p)}\left(0\right)/p! \tag{14}$$

The risk premium $\mathbf{RP}(X)$ is now an infinite sum of elementary premia each of which is the product of two terms: the pth order centered moment of the random variable X, which is nothing but a *quantity of risk*, and $\mathcal{D}^{(p)}\left(0\right)/p!$ which characterizes the magnitude of risk aversion. If a rational decision maker is averse to any nth-degree increase in risk $-$*i.e.* if he prefers X to Y whenever Y is a nth-degree increase in risk over $X$$-$, then $sgn(\mathcal{D}^{(p)}\left(0\right)) = (-1)^{p}$ (See Jouini *et al.* 2013.) his preferences may be represented by the following functional:

$$\mathcal{U}(X) = \mathbf{E}[X] - \textstyle\sum_{p=2}^{+\infty} (-1)^{p} \mathcal{A}_{p}\mathbf{E}\left[\left(X - \mathbf{E}\left[X\right]\right)^{p}\right] \tag{15}$$

where $\mathcal{A}_p \overset{def}{=} (-1)^{p} \mathcal{D}^{(p)}\left(0\right)/p! > 0$. The above equation may be viewed as a *theoretical grounding of the multimoment approach of the Capital Asset Pricing Model.*

The last definition of the consistency between risk and risk aversion states that preferences should be represented by a monotonous function of a measure of risk *à la* Artzner *et alii* (1997).[12] The invariance of risk premia by translation is a desirable feature of a coherent measure of risk. Unfortunately, theories of decision making under risk do not generally imply such a result. In contrast, this theory of disappointment does. Indeed one may set:

$$\mathbf{r}(X) = -\mathcal{U}(X) = -\mathbf{E}[X] + \int_{a}^{b} \mathcal{D}(x - \mathbf{E}[X]) dF_X(x)$$

and, since $\mathcal{D}(.)$ is convex, $\mathbf{r}(X)$ appears as *a convex measure of risk* of X in the sense of Föllmer and Schied (2002).[13] Hence, the above equality allows for grounding a convex measure of risk on a theory of the behaviour of economic

[11] See *e.g.* K. Lange *Applied Probability* Springer (2003).

[12] Who define a measure of risk as the minimum amount of money which must be added to a risky portfolio (or a risky position) to make the risk incurred by the owner of the portfolio (or the holder of the position) acceptable by a risk controller.

[13] The proof of this statement may be found in Chauveau Th. and St.Thomas (2015): *Valuing non-quoted CDS with consistent default probabilities*, unpublished manuscript.

agents towards risk. *The risk controller* is then assumed to be *a risk-averse rational decision maker with constant marginal utility.*

To sum up, we have developed a fully choice-based theory of disappointment which clearly may give rise to many applications in Finance. However, the assumption of constant marginal utility is obviously too restrictive and, consequently, we now turn to the general case of variable marginal utility. The corresponding theory will be developped in the next section, including a new axiomatic which will be nothing but a slightly modified version of the present one.

4 *A general theory of disappointment with lottery dependent utility*

We go back to the framework developed for the presentation of the generalized EU theory. A rational risk averse investor will be endowed with a likely utility function $\mathbf{u}(.)$ and we again use this result to set a new axiomatic over utilities.

4.1 The axiomatic

The two first axioms are still those of the generalized EU theory (See axioms I and II above). However, the third axiom now reads:

Axiom III'. (linearity of certainty equivalents of u-lotteries belonging to $\mathfrak{U}_{\overline{\mathbf{u}}}$)

The certainty equivalent of the $(\lambda, 1-\lambda)$*-mixing of two* $\mathbf{u}$*-lotteries which exhibit the same expected value is the corresponding convex combination of their certainty equivalents, what formally reads:*

$$\forall U_1, U_2 \in \mathfrak{U}_{\overline{\mathbf{u}}},\ \forall \lambda \in [0,1],\ \mathfrak{c}(\lambda U_1 \oplus (1-\lambda) U_2) = \lambda \mathfrak{c}(U_1) + (1-\lambda)\mathfrak{c}(U_2)$$

where $\mathfrak{U}_{\overline{\mathbf{u}}} = \{U \in \mathfrak{U} \,|\mathbb{E}[U] = \overline{\mathbf{u}}\}$

Finally, Axioms I, II and III' imply the following results, which are analogous to those already presented in the previous section.

Proposition 10. (lottery-dependent expected utility representation theorem for $\succsim_{\mathrm{u}}$ over $\mathfrak{U}$). *Under Axioms I, II and III', the preorder of preferences of a risk-averse decision maker –labelled* $\succsim_{\mathrm{u}}$*– can be represented over* $\mathfrak{U}$ *by the following lottery-dependent functional*:

$$\mathcal{V}(U) \overset{def}{=} \int_0^1 \mathfrak{v}_{\mathbb{E}[U]}(z) dG_U(z) \tag{16}$$

where $\mathfrak{v}_{\mathbb{E}[U]}(.)$ *is a continuous and increasing function mapping* $[0,1]$ *on to itself which is defined up to an affine and positive transformation.*

Proof. It is the same as the proofs given in Section 3 for Propositions 5 to 7.$\square$

The conditions of normalization will now read:

$$\mathfrak{v}_{\mathbb{E}[U]}(\mathbb{E}[U]) = \mathbb{E}[U] \quad and \quad \mathbb{E}[U]\,\mathfrak{v}_{\mathbb{E}[U]}(1) + (1-\mathbb{E}[U])\,\mathfrak{v}_{\mathbb{E}[U]}(0) = \mathfrak{c}\left(U_{\mathbb{E}[U]}\right) \tag{17}$$

and, here again, $\mathfrak{v}_{\mathbb{E}[U]}(.)$ is unique and well defined.

One may now assume that risk premia are *translation-invariant when they are valued in utils*, *i.e.* that the following property is met:

$$\mathbb{RP}(U + \Delta u) = \mathbb{RP}(U) \tag{18}$$

where: $\mathbb{RP}(U) = \mathbb{E}\left[U\right] - \int_0^1 \mathfrak{v}_{\mathbb{E}[U]}(z) dG_U\left(z\right)$.

Using similar arguments as those already developed in Section 3, one can show that the invariance by translation property implies that the functional representing the decision maker's preferences may now be rewritten as:

$$\mathcal{V}(U) = \mathbb{E}[U] - \int_0^1 \mathcal{D}(z - \mathbb{E}[U]) dG_U\left(z\right) \tag{19}$$

where $\mathcal{D}(.)$ is a *disappointment function*.

A *subjective risk premium* will then be defined as the following difference:

$$\mathbb{RP}\left(U\right) \overset{def}{=} \mathbb{E}\left[U\right] - \mathcal{V}(U) = \mathbb{E}\left[U\right] - \mathfrak{c}(U) \tag{20}$$

A *subjectively risk averse* decision-maker will prefer U to V whenever V is a mean preserving spread of U. A necessary and sufficient condition for subjective risk aversion to occur is that any subjective risk premium is positive unless U is degenerate, *i.e.*

$$\int_0^1 \mathcal{D}(z - \mathbb{E}[U]) dG_U\left(z\right) = \mathbb{E}\left[\mathcal{D}(U - \mathbb{E}[U])\right] > 0 \Longleftrightarrow U \textit{ is not degenerate} \tag{21}$$

or, equivalently, that $\mathcal{D}(.)$ is strictly *convex* (*i.e.* $\mathcal{D}''(.) > 0$ if $\mathcal{D}(.)$ is twice continuously derivable).

The above results may be rewritten in terms of (monetary valued) random outcomes. For instance, Equation (19) may be rewritten as:

$$\mathcal{U}(X) = \mathbf{E}[\mathbf{u}\left(X\right)] - \int_a^b \mathcal{D}(\mathbf{u}\left(x\right) - \mathbf{E}[\mathbf{u}\left(X\right)]) dF_X(x) \tag{22}$$

and a decision maker is rational if his preferences are represented by a functional $U(.)$ defined by (22) where $\mathcal{D}(.)$ is a disappointment function. All these transpositions are straightforward. Equation (22) may be viewed as *defining the functional of LS-models.*

Moreover, if $\mathcal{D}(.)$ is an infinitely derivable disappointment function, the subjective risk premium may be rewritten as:

$$\mathbb{RP}\left(\mathbf{u}\left(X\right)\right) = \textstyle\sum_{p=2}^{+\infty} \mathbf{E}\left[\left(\mathbf{u}\left(X\right) - \mathbf{E}\left[\mathbf{u}\left(X\right)\right]\right)^p\right] \dfrac{\mathcal{D}^{(p)}\left(0\right)}{p!} \tag{23}$$

and the elementary premia are now the contributions of the variance, the skewness, the kurtosis ... of the *utility* of a lottery to the total risk premium which is demanded by a decision maker.

Finally the preferences of a rational risk-averse decision maker can be represented by the following functional:

$$\mathcal{U}(X) = \mathbf{E}[\mathbf{u}(X)] - \textstyle\sum_{p=2}^{+\infty} (-1)^p \mathcal{A}_p \mu_p (\mathbf{u}(X)) \tag{24}$$

where $\mu_p(\mathbf{u}(X)) \overset{def}{=} \mathbf{E}[(\mathbf{u}(X) - \mathbf{E}[\mathbf{u}(X)])^p]$. In the case when $\mathbb{RP}(U)$ is reduced to its first term, $\mathcal{U}(X)$ is equal to the expected utility *minus* a penalty which is equal to the variance of the utility of the lottery, *i.e.*:

$$\mathcal{U}(X) = \mathbf{E}[\mathbf{u}(X)] - \mathcal{A}\mathbf{Var}[\mathbf{u}(X)] \tag{25}$$

Such a result is connected with a conjecture of Allais (1979) who argued that a positive theory of choice should contain two basics elements: (i) the existence of a cardinal utility function which is independent of risk attitudes and (ii) a valuation functional of risky lotteries which depends on the second moment of the probability distribution of uncertain utility. If constant marginal utility is assumed, the above approach comes down to mean-variance analysis. Mean-variance analysis can thus be viewed as a particular case of the LS-theory.[14]

However EU theory, and, consequently mean-variance analysis, is often violated by experiments and no general agreement has yet been found about the explaining power of its challengers, *i.e.* Non-EU theories. Hence it is interesting to point out that, because of its flexibility, the functional (22) is compatible with many of the anomalies of financial theory.[15] However, this issue is beyond the scope of this article.

The above results (See Propositions 9 and 10) have been obtained from a set of axioms dealing with preferences over **u**-lotteries. Note that they can also be obtained differently, from a set of axioms dealing with preferences over **m**-lotteries (See Appendix 2).

Finally a functional such as $\mathcal{U}(.)$ which is defined by (22) and which meets the above requirements will be called, from now on, a LS-*functional* and will characterize LS-*models*.

4.2 Overview of the related literature

We now give a presentation of some links existing between LS-models and some other disappointment models. As a preliminary, we emphasize that LS-models are close to models *à la* Loomes and Sugden (1986) whose functional reads:

$$\mathcal{U}(X) = \mathbf{E}[\mathbf{u}(X)] - \int_a^b \mathcal{D}(\mathbf{u}(x) - \overline{\mathbf{u}}) dF_X(x)$$

since one may set $\overline{\mathbf{u}} = \mathbf{E}[\mathbf{u}(X)]$ in the above functional, to make $\mathcal{U}(.)$ become lottery-dependent.

[14] EU theory obviously provides the same result, but at the expense of an additional assumption about the distribution of asset returns or the investor's utility function.

[15] See *e.g.* Chauveau Th. and N. Nalpas. 2005. Disappointment, Pessimism and the Equity Premia, *Cahier de Recherche*, Toulouse Business School Working Paper, septembre.

Disappointment or elation may also be measured somewhat differently: Delquie and Cillo (2006) use all the outcomes of the lottery; Grant and Kajii (1998) adapt the setting of the rank-dependent expected utility model (Quiggin (1982) among others) to highlight the dependence on the best possible outcome; Jia *et alii* (2001) generalize Bell's (1985) approach and advocate the use of its expected value. Indeed, they consider the expected value of the lottery as the reference point for measuring disappointment. Their preference functional can be defined as:

$$\mathcal{U}_{JDB}\left(X\right)=\int_{a}^{b}\left(1+d\mathbf{1}_{[x<\mathbf{E}(X)]}-e\mathbf{1}_{[x>\mathbf{E}(X)]}\right)xdF_{X}(x) \tag{26}$$

where d and e are two positive parameters. The above functional is nothing but a particular case of (10).[16]

LS-models can also be viewed as particular case of lottery dependent utility (henceforth LDU) models which were first developed by Becker and Sarin (1987). The preference functional of a LDU model is then:[17]

$$\mathcal{U}_{LDU}\left(X\right)=\textstyle\sum_{k=1}^{K}p_{k}v\left(h(X),x_{k}\right) \tag{27}$$

where $v\left[.,.\right]$ is a function defined over $[a,b]\times\mathbb{R}^{+}$ and whose values belong to $[0,1]$ and where $h(.)$ is a function defined over $\mathfrak{X}$ and whose values belong to R. The functional (27) can be derived from three axioms which have been provided by Becker and Sarin (1987): total ordering, continuity and monotonicity. Their first two axioms are those of EU theory and the third one is nothing but the SSD dominance principle. However, as pointed out by Starmer (2000) "the basic model is conventional theory for minimalists as, without further restriction, it has virtually no empirical content."[18] Finally, almost any non-EU model can be viewed as a LDU model, once an appropriate functional form of $v(.)$ has been chosen.

Becker and Sarin then particularize their model assessing $h\left(X\right)$ to be linear with respect to the probabilities p_k, *i.e.* they set $h(X)=\sum_{k=1}^{K}h_{k}p_{k}$ and they define a function $H(.)$ such that $H\left(x_{k}\right)=h_{k}$. To sum up, the authors set:

$$h(X)=\textstyle\sum_{k=1}^{K}H\left(x_{k}\right)p_{k}=\mathbf{E}\left[H\left[X\right]\right] \tag{28}$$

and the new model then belongs to a subset of LDU models called lottery-dependent expected utility models (henceforth LDEU models). The functions $h(.)$, or $H(.)$, may be chosen arbitrarily but they have to be specified before testable implications of the model be derived. As a consequence, LDEU models are not choice-based.

Schmidt (2001) considers somewhat more general models called "lottery-dependent convex utility models" (henceforth LDCU models). A condition less restrictive than (28) is fulfilled by LDCU models. It reads:

$$h(X_{i})=\lambda \text{ and } \alpha_{i}\geq 0 \text{ and } \textstyle\sum_{i=1}^{N}\alpha_{i}=1\Rightarrow h\left(\sum_{i=1}^{N}\alpha_{i}X_{i}\right)=\lambda \tag{29}$$

[16] To see this point, just set: $\mathbf{u}\left(x\right)=x$ and $u_{\mathbf{E}[X]}(x)=\left(1+d\mathbf{1}_{[x<\mathbf{E}[X]]}-e\mathbf{1}_{[x>\mathbf{E}[X]]}\right)x$

[17] To make short only simple lotteries are taken into account.

[18] Starmer: *Developments in Non Expected Utility Theory*, JEL, p. 345.

Four axioms are necessary to develop this class of models. The first two axioms (total ordering and continuity) are, again, those of EU theory. The author then substitutes for the independence axiom two new axioms: the first one, called the lottery dependent independence axiom, states that the independence property is met over any subset $\mathfrak{X}_\lambda$ of lotteries fulfilling (29). However, to derive LDCU models, $\mathcal{U}(X)$ has to be linear in every subset $\mathfrak{X}_\lambda$ for all λ. A linear $\mathcal{U}(.)$ is obtained *iff* there exists a sequence of functions $\{\varphi_\lambda, \lambda \in [\mathcal{U}(\delta(0)), \mathcal{U}(\delta(1))]\}$ where $\varphi_\lambda : \mathbb{R} \rightarrow \mathbb{R}^+$ is continuous and increasing so that:

$$\forall \lambda \in [\mathcal{U}(\delta(0)), \mathcal{U}(\delta(1))], \mathcal{U}(X) = \varphi_\lambda [u_\lambda(X)] \ \ if \ X \in \mathfrak{X}_\lambda$$

This result is guaranteed by an additional axiom which is called the linearity axiom and which enables him to select one particular function $\mathcal{U}(.)$ from all the candidates. In contrast to LS-models, Schmidt's approach is not fully choice-based since using the axioms implies that the function $h(.)$ be known on *a priori* grounds.[19]

Finally, it must be underlined that Gul (1991) developed an *implicit expected utility model of disappointment* where the certainty equivalent of the lottery plays the role of reference level. It is fully axiomatized and some attempts have been made to use this kind of preferences in an asset pricing model (See *e.g.* Ang et alii 2005 and Routledge and Zin 2004). However, Gul's theory does not guarantee that the utility function which is derived from their axiomatic is the most likely to be that of the decision maker. Moreover it is not endowed with the invariance by translation property. In contrast, it is well endowed with an elicitation property which is due to an extension (see Abdellaoui M. and H. Bleichrodt 2007) of the trade-off method (Wakker and Deneffe 1996).

LS-models are also endowed with the elicitation property. Indeed, the likely utility function can be elicited thanks to a method which is also parameter-free and requires no assumption about utility nor disappointment aversion. To spare space, this point is developed in Appendix 3.

5 *Concluding remarks*

In this paper, it has been shown that some important information about the preferences of a decision maker can be gleaned from comparing his preferences over lotteries and the preorders induced by stochastic dominances over random utilities. This information is neglected in other theories of decision making under risk. Consequently they cannot guarantee that the utility function which is derived from their axiomatic is the most likely to be that of the decision maker.

It has been shown that there exists one utility function which is best in accordance with the decision maker's preferences in that it induces a SSSD relation over lotteries which is the closest to the original preorder of preferences.

[19] This is probably because the utility function has not been defined independently of the setting of the axioms.

It has been called the likely utility function. It is consistent and, consequently no violations of SSSD occur, what is a logical requirement. It is nothing but the superior enveloppe of the consistent concave or convex utility functions.

An original theory of disappointment, where the likely utility function is defined before the axiomatic is set, has been developed. It remains close to that of Loomes and Sugden (1986), although preferences are now lottery-dependent. Its axiomatic includes the two first axioms of EU theory and a linearity axiom for the certainty equivalents of lotteries which exhibit the same (likely) expected utility. An expected utility representation of preferences has been derived from this axiomatic. The assumption of the translation invariance of risk premia then allows for a functional which is that of Loomes and Sugden except that the reference level is now the decision maker's wealth expected utility.

LS-models are endowed with many interesting properties which have been presented above. Among them one must be emphasized: the consistency between the definitions of risk and risk aversion and, in particular, the compatibility of the assumed behaviour of the decision maker with the definition of a coherent measure of risk.

Finally, the above theory of disappointment may be used to understand the behaviour of risk-averse agents on the financial markets. It is compatible with the so-called "financial anomalies" or the equity premium puzzle (Mehra and Prescott 1985).[20] What is more, under the assumption of constant relative risk aversion(s), one can easily implement the above approach to value any financial asset. An example of this valuation for CDS's is being developed.[21]

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6 *Appendix 1. The case of piecewise concave/ convex likely utility functions*

In this subsection, we shall allow for utility functions which are not concave/convex over the whole interval $[a,b]$. We set the following definition:

Definition 6 (w_0-utility function). Let $u_a(.)$ ($u_b(.)$) be a C1 utility function defined over $[a,w_0]$ ($[w_0,b]$) and let $u(.) : [a,b] \longrightarrow [0,1]$ be defined by:

$u(x) = \lambda(w_0)u_a(x)$ for $x\in[a,w_0]$ *and*

$u(x) = \lambda(w_0)+(1-\lambda(w_0))u_b(x)$ for $x\in[w_0,b]$

where $\lambda(w_0) = u_b'(w_0)/(u_a'(w_0) + u_b'(w_0)) \in [0,1]$. Then function $u(.)$ will be called the w_0-utility function canonically associated with $u_a(.)$ and $u_b(.)$. *Conversely, if $u(.)$ is a w_0-utility function over $[a,b]$, then there exists a unique C1 utility function defined over $[a,w_0]$ and a unique C1 utility function defined over $[w_0,b]$ such that $u(.)$ is canonically associated with $u_a(.)$ and $u_b(.)$. The latter functions are given by:*

$u_a(x)=u(x)/u(w_0)$ for $x\in[a,w_0]$ *and*

$u_b(x)=u(w_0)+(u(x)-u(w_0))/(1-u(w_0))$ for $x\in[w_0,b]$

If $u_a(.)$ and $u_b(.)$ are both concave (both convex), then $u(.)$ is concave (convex). If $u_a(.)$ is concave (convex) and $u_b(.)$ convex (concave) then function $u(.)$ is piecewise concave or convex. When a w_0-utility function is piecewise concave or convex, then it has an inflection point at $(w_0, u(w_0))$. A geometrical illustration of the above definitions is given on Figures 2A and 2B.

The following proposition may now be substituted for Proposition 3.

Proposition 11. (*likely w_0-utility function*). *Let w_0 be an arbitrary level of wealth belonging to* $]a,b[$. *There exists a unique consistent C1 w_0-utility function $u(.)$ mapping* $[a,b]$ *on to* $[0,1]$ *which is such that the preorder $\succcurlyeq_2^{\mathbf{u}}$ is the closest to the preorder of preferences $\succsim$ among the preorders induced by consistent w_0-utility functions over* $[a,b]$. *It is the w_0-utility function canonically associated with the likely utility functions $\mathbf{u}_a(.)$* and $\mathbf{u}_b(.)$ *which are defined over* $[a, w_0]$ and $[w_0,b]$. *It is piecewise concave/convex. It will be called, from now on, the likely w_0-utility function of the decision maker.*

Proof. It is similar to the proof of Proposition 3.□

In the above discussion, we have focused on w_0-utility functions because the investor's behaviour may differ, as suggested by Kahneman and Tversky (1979), when he faces gains or losses, *i.e.* when his ex-post wealth X is higher or lower than his initial (certain) wealth w_0. Gains (losses) occur when $X(\omega)$ is higher (lower) than w_0. However, the above analysis could clearly be extended to functions whose graph includes N successive concave and convex sections. However, such an extension is beyond the scope of this article.

7 *Appendix 2. An alternative axiomatic*

Consequences will now be valued in monetary units. As in Section 3, Axioms 1 and 2 will be set: they are clearly equivalent to Axioms I and II. In contrast, Axiom 3 is not equivalent to Axiom III' and, consequently, a new axiom will now be set:

Axiom 3'. The utility of the certainty equivalent of the $(\lambda,1-\lambda)$-*mixing of two* $\boldsymbol{m}$-*lotteries which exhibit the same expected utility is the corresponding convex combination of the utilities of their certainty equivalents, what formally reads*:

$\forall X_1, X_2 \in \mathbf{X}_{\overline{\mathbf{u}}}, \forall \lambda \in [0,1], \mathbf{u}(\mathbf{c}(\lambda X_1 \oplus (1-\lambda) X_2)) = \lambda \mathbf{u}(\mathbf{c}(X_1)) + (1-\lambda)\mathbf{u}(\mathbf{c}(X_2))$
where $\mathbf{X}_{\overline{\mathbf{u}}} = \{X \in \mathfrak{X} \mid \mathbf{u}(x) = \overline{\mathbf{u}})$

Axiom 3' is clearly equivalent to Axiom III' and, consequently, setting Axioms 1, 2 and 3' is equivalent to setting Axioms I, II and III'. As a consequence, we get the following result:

Proposition 12. (*lottery-dependent expected utility representation theorem for $\succsim$ over X*).*s Under Axioms 1,2, and 3', the preorder of preferences $\succsim$ of a risk averse decision maker can be represented over X by the following lottery-dependent functional:*

$$\mathcal{U}(X) \overset{def}{=} \int_a^b \mathfrak{u}_{\mathbf{E}[\mathbf{u}(X)]}(x) dF_X(x) \tag{30}$$

where $\mathfrak{u}_{\mathbf{E}[\mathbf{u}(X)]}(.)$ is a continuous and increasing function mapping $[a,b]$ *on to* $[0,1]$ *and which meets the normalization conditions* (17).

Proof. Proposition 12 is clearly equivalent to Proposition 10.□

8 *Appendix 3. The elicitation property*[22]

As shown in the following subsection, the likely utility function can be elicited thanks to an elicitation method which is parameter-free and requires no assumption about utility nor disappointment aversion.

8.1 Preliminary definitions and results

Recall that, unlike EU models, LS-models are not endowed with a global independence property. In particular, some couples of indifferent lotteries ($(X_1, X_2) \in \mathfrak{X} \times \mathfrak{X}$ with $X_1 \sim X_2$) are such that $\lambda X_1 \oplus (1-\lambda) X_2 \nsim X_1$ for some values of $\lambda \in [0,1]$. However, there also exist, in these models, some couples of indifferent lotteries which do exhibit the betweenness property: they will be called, from now on, strongly indifferent lotteries.

Definition 7. (*strong indifference*). *Two lotteries X_1 and X_2 are strongly indifferent iff they meet the following requirement:*

$$\forall \lambda \in [0,1],\ \lambda X_1 \oplus (1-\lambda) X_2 \sim X_1 \tag{31}$$

Strong indifference can be characterized in the following way:

Proposition 13. In LS-models, two lotteries X_1 and X_2 are strongly indifferent iff they exhibit the same certainty equivalent and the same expected utility, what formally reads:

$$X_1 \approx X_2 \Longleftrightarrow \mathbf{c}(X_1) = \mathbf{c}(X_2) \ \ and \ \ \mathbf{E}[\mathbf{u}(X_1)] = \mathbf{E}[\mathbf{u}(X_2)]$$

Proof. It is given in Appendix 3.□

The binary relation "X_1 and X_2 are strongly indifferent" will be labelled "$X_1 \approx X_2$". It is obviously reflexive and symmetric. From Proposition 13 we get that it is also transitive and, consequently, it is an equivalence relation over $\mathfrak{X}$. Finally, note that strong indifference implies indifference in the usual sense which will be called, from now on, weak indifference. A related new concept needs now to be introduced: that of strong equivalents.

Definition 8. (*strong equivalents*). *Let X_1 be an arbitrary lottery and let*:

$$\underline{X}_p^x \overset{def}{=} [a, x; 1-p, p] \ \ ; \ \ \overline{X}_q^y \overset{def}{=} [y, b; q, 1-q]$$

where $x,y \in]a,b[$. Then if X and $\underline{X}_p^x$ ($\overline{X}_q^y$) are strongly indifferent, then $\underline{X}_p^x$ ($\overline{X}_q^y$) is the left (right) strong equivalent of X.

The above definition will make sense only if any lottery is endowed with a unique couple of strong equivalents. As indicated in the next proposition, this happens to be the case.

Proposition 14. In LS-models, a lottery has exactly one left and one right strong equivalent.

Proof. It is given in Appendix 3.□

[22] This subsection is but a rewriting of: Chauveau Th., N. Nalpas [2010]. *Disappointment models: an axiomatic approach*, CES workingpaper, 2010.102

8.2 The elicitation property

We now turn to the elicitation property. The first step of the argument is as follows: let $\overline{X}_q^y$ be the right strong equivalent of $\underline{X}_p^x$ *i.e.* let:

$$\overline{X}_q^y \approx \underline{X}_p^x$$

Then, the difference between the expected utility of $\underline{X}_p^x$ and that of $\underline{X}_q^y$ is $1-q$.[23] Indeed, we get that:

$$\mathbf{E}[\mathbf{u}(\underline{X}_p^x)] - \mathbf{E}[\mathbf{u}(\underline{X}_q^y)] = \mathbf{E}[\mathbf{u}(\overline{X}_q^y)] - \mathbf{E}[\mathbf{u}(\underline{X}_q^y)] = ((1-q)+qu(y)) - qu(y) = 1-q$$

The second step consists in defining a sequence of binary lotteries, $\{\underline{X}_{p_n}^{x_n}\}_{n\in\mathbb{N}}$ as indicated below:

$$x_0 = w \ ; \ p_0 = \pi \ \text{ and } \ \overline{X}_{p_{n+1}}^{x_{n+1}} \approx \underline{X}_{p_n}^{x_n} \tag{32}$$

Note that, if $\overline{X}_q^y$ is the right strong equivalent of $\underline{X}_p^x$, then $y < x$. As a consequence, $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. Moreover, it is such that the difference between the expected utilities of two consecutive binary lotteries, $\underline{X}_{p_n}^{x_n}$ and $\underline{X}_{p_{n+1}}^{x_{n+1}}$, is equal to the second weight $(1-p_{n+1})$ of the right strong equivalent of $\underline{X}_{p_n}^{x_n}$, what formally reads:

$$\mathbf{E}[\mathbf{u}(\underline{X}_{p_n}^{x_n})] - \mathbf{E}[\mathbf{u}(\underline{X}_{p_{n+1}}^{x_{n+1}})] = 1 - p_{n+1}$$

As a consequence, we get that:

$$\pi u(w) - \mathbf{E}[\mathbf{u}(\underline{X}_{p_n}^{x_n})] = \mathbf{E}[\mathbf{u}(\underline{X}_{p_0}^{x_0})] - \mathbf{E}[\mathbf{u}(\underline{X}_{p_n}^{x_n})] = \sum_{i=1}^n (1-p_i)$$

and the expected utility of the initial lottery is the sum of the expected utility of any element of the sequence and of the accumulation of the second weights of the right strong equivalents, what formally reads:

$$\pi u(w) = \mathbf{E}[\mathbf{u}(\underline{X}_{p_0}^{x_0})] = \mathbf{E}[\mathbf{u}(\underline{X}_{p_n}^{x_n})] + \sum_{i=1}^n (1-p_i)$$

Alternatively, one could consider the following sequence of binary lotteries:

$$y_0 = w \ ; \ q_0 = \pi \ ; \ \underline{X}_{q_{n+1}}^{y_{n+1}} \approx \overline{X}_{1-q_n}^{y_n} \tag{33}$$

The elements of the sequence $\{\overline{X}_{1-q_n}^{y_n}\}_{n\in\mathbb{N}}$ are endowed with the following property:

$$\mathbf{E}[\mathbf{u}(\underline{X}_{q_{n+1}}^{y_{n+1}})] - \mathbf{E}[\mathbf{u}(\underline{X}_{q_n}^{y_n})] = 1 - q_{n+1} \Longrightarrow \pi\mathbf{u}(w) = \mathbf{E}[\mathbf{u}(\underline{X}_{q_n}^{y_n})] - \sum_{i=1}^n (1-q_i)$$

From now on, the sequences $\{\underline{X}_{p_n}^{x_n}\}_{n\in\mathbb{N}}$ and $\{\overline{X}_{1-q_n}^{y_n}\}_{n\in\mathbb{N}}$, will be called the canonical sequences generated by (w,π). The first (second) one is the left (right) canonical sequence. As shown below, they respectively converge, in LS-models, towards $\delta(0)$ or $\delta(1)$.

[23] Recall that $\underline{X}_q^y \overset{def}{=} [a, y; 1-q, q]$

Proposition 15. *Let* $(w,\pi)\in]a,b[\times]0,1[$. *Consider the canonical sequences of binary lotteries generated by* (w,π). *Then, in LS-models where decision makers are disappointment averse, the left (right) canonical sequence is decreasing*[24] *(increasing*[25]*) and converges towards* $\delta(a)$[26] $(\delta(b)$[27]$)$. *Moreover, we get the following equalities*:

$$\mathbf{u}(w) = (\sum_{i=1}^{\infty}(1-p_i))/\pi = (1-\sum_{i=0}^{\infty}(1-q_i))/\pi \qquad (34)$$

Proof. It is given in Appendix 3.□

Finally, in LS-models, the set of lotteries $\mathfrak{X}$ is well endowed with the elicitation property, *i.e.* the value of $u(w)$ can be elicited with as much accuracy as desired for any outcome $w\in]a,b[$. Indeed, one can choose an arbitrary probability $\pi\in]0,1[$ and build, from the answers of a decision maker facing lotteries of the $\underline{X}_p^x$ type and/or of the $\overline{X}_q^y$ type, the two canonical sequences generated by (w,π). An accurate ranging of $u(w)$ should be obtained since we have the inequalities:

$$0 < \sum_{i=1}^{n}(1-p_i))/\pi \leq \mathbf{u}(w) \leq (1-\sum_{i=1}^{n}(1-q_i))/\pi \qquad (35)$$

Once $u(w)$ has been elicited, the aversion coefficients of (24) can be elicited in a standard way.

9 *Appendix 4. Proofs*

9.1 Proof of Proposition 1.

Let $Y_i{=}u(X_i)$ for $i{=}1{,}2$. By definition of SSSD, it is equivalent to state:
(a) $X_1 \succcurlyeq_2^u X_2$, or:
(b) $U_1 \succcurlyeq_2 U_2$, or, equivalently,
(c) $\int_0^v [F_{U_1}(t) - F_{U_2}(t)]dt \leq 0$ for $v \in [0{,}1]$.
Now, since we have:
$\int_0^v [F_{U_1}(t) - F_{U_2}(t)]dt = \int_a^{u^{-1}(v)} [F_{X_1}(x) - F_{X_2}(x)]u'(x)dx.$ (I)
condition (c) is equivalent to the following one:
$\int_a^z [F_{X_1}(x) - F_{X_2}(x)]u'(x)dx \leq 0$ for any $z\in[a,b]$.
□

9.2 Proof of Proposition 2.

As a preliminary, recall that $u(.)$ is more concave than $v(.)$ *iff* $u \circ v^{-1}(.)$ is concave *i.e.* if there exists a concave function $g(.)$ mapping $[v(a), v(b)]$ on to $[u(a), u(b)]$ and such that: $u(x) = g \circ v(x)$ for $x \in [a,b]$.

[24] That is $\underline{X}_{p_m}^{x_m}$ is preferred to $\underline{X}_{p_n}^{x_n}$ if $n \geq m$

[25] That is $\overline{X}_{q_n}^{y_n}$ is preferred to $\overline{X}_{q_n}^{y_n,b}$ if $n \geq m$

[26] Equivalently, one can say that $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of real numbers converging towards a and that $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers converging towards 1.

[27] Equivalently, one can say that $\{y_n\}_{n\in\mathbb{N}}$ is an increasing sequence of real numbers converging towards b and that $\{q_n\}_{n\in\mathbb{N}}$ is a sequence of real numbers converging towards 1.

Since, by assumption, $u(.)$ and $v(.)$ are increasing and C1, $g(.)$ is also increasing and C1 and we get that $g'(.) > 0$. The rest of the proof is grounded on the following calculations: let

$$\Delta(z)\overset{def}{=}\int_a^z u'(x)(F_{X_1}(x) - F_{X_2}(x))dx$$
$$\Phi(z)\overset{def}{=}\int_a^z v'(x)(F_{X_1}(x) - F_{X_2}(x))dx$$
$$G(z)\overset{def}{=}g'(v(z))$$

Then, $\Delta(z)$ may be rewritten as:

$$\Delta(z)=\int_a^z g'(v(x))v'(x)(F_{X_1}(x) - F_{X_2}(x))dx = \int_a^z G(x)d\Phi(x)$$

Integrating by parts yields:

$$\Delta(z) = G(z)\Phi(z) - \int_a^z \Phi(x)dG(x) \tag{36}$$

Recall that $G(x) > 0$, that $G(.)$ is decreasing and that $v'(x) > 0$. Hence if $\Phi(z) \overset{def}{=} \int_a^z v'(t)(F_{X_1}(t) - F_{X_2}(t))dt$ has a constant sign for any $z\in[a, b]$, $\Delta(z)$ is endowed with that sign. In particular, we get the following result, which holds for any $z\in[a, b]$:

$$\int_a^z v'(t)(F_{X_1}(t) - F_{X_2}(t))dt < 0 \ \Rightarrow \int_a^z u'(x)(F_{X_1}(x) - F_{X_2}(x))dx = \Delta(z) < 0$$

or:

$$X_1 \succcurlyeq_2^v X_2 \Rightarrow X_1 \succcurlyeq_2^u X_2 \quad \Leftrightarrow \quad \mathbb{X}_2^v \subset \mathbb{X}_2^u$$

and, as a consequence:

$$\mathbb{X}_2^{v-} \subseteq \mathbb{X}_2^{u-} \ and \ \mathbb{X}_2^{v+} \subseteq \mathbb{X}_2^{u+}$$

□

9.3 Proof of Proposition 3.

Let $\mathbb{U}$ denote the subset of *utility functions which are, by assumption, increasing, derivable and concave or convex* (henceforth c/c). Let the subset of concave (convex) utility functions be labelled $\mathbb{U}^a$ ($\mathbb{U}^x$). We have:

$$\mathbb{U} = \mathbb{U}^a \cup \mathbb{U}^x \ and \ \mathbb{U}^a \cap \mathbb{U}^x = \{\mathbf{f}(.)\}$$

where $\mathbf{f}(.)$ is the affine function defined by $\mathbf{f}(x) = (x-a)/(b-a)$. *Note that the graph of any concave (convex) utility function is above (below) that of* $\mathbf{f}(.)$.

Let $\mathbb{U}_{\mathrm{I}}$ ($\mathbb{U}_{\mathrm{C}}$) be the subset of inconsistent (consistent) c/c utility functions and $\mathbb{U}^a_{\mathrm{C}}$ ($\mathbb{U}^a_{\mathrm{I}}$, $\mathbb{U}^x_{\mathrm{C}}$, $\mathbb{U}^x_{\mathrm{I}}$) be the subset of concave consistent (concave inconsistent, convex consistent, convex inconsistent) utility functions. Two cases may occur, according to the fact that the (standard) SSD dominance property is violated or not.

A. We first assume that *the (standard) SSD dominance property is not violated*. As a consequence, there exists at least one concave function which is consistent. It is the affine function $\mathbf{f}(.)$. Hence: $\mathbb{U}^a_{\mathrm{C}} \neq \varnothing$; $\mathbb{U}_{\mathrm{C}} = \mathbb{U}^a_{\mathrm{C}} \cup \mathbb{U}^x_{\mathrm{C}}$. Moreover, we have: $\mathbb{U}^x_{\mathrm{I}} = \varnothing$ and $\mathbb{U}_{\mathrm{I}} = \mathbb{U}^a_{\mathrm{I}}$. Indeed, if $\mathbb{U}^x_{\mathrm{I}}$ were not empty there would exist some inconsistent convex functions whose graph would be below that of $\mathbf{f}(.)$.

A trivial subcase is when $\mathbb{U}^a_{\mathrm{C}} = \{\mathbf{f}(.)\}$. Proposition 3 is then clearly valid.

We now leave aside this trivial case and we assume that $\mathbb{U}^a_{\mathrm{C}}$ includes at least one strictly concave utility function. *The superior enveloppe of the consistent c/c utility functions* is then defined by:

$$\mathbf{u}(.) = \sup_{u \in \mathbb{U}_{\mathrm{C}}} (u) = \sup_{u \in \mathbb{U}^a_{\mathrm{C}}} (u)$$

Let $epi(u)$ be the epigraph of u. Then all the functions belonging to $\mathbb{U}^a_{\mathrm{C}}$ are concave and, from standard convex analysis, we get that:

$$epi(\mathbf{u}) = \cap_{u \in \mathbb{U}_{\mathrm{C}}} epi(u)$$

and, consequently, that $\mathbf{u}(.)$ is concave. The epigraph of any consistent utility function then includes $epi(\mathbf{u})$ *i.e.*:

$$u \in \mathbb{U}_{\mathrm{C}} \Rightarrow epi(\mathbf{u}) \subseteq epi(u)$$

Finally, note that $\mathbf{u}(.)$ is obviously increasing and normalized.

Now we want to prove that $\mathbf{u}(.)$ is also consistent and derivable. To see this, consider a pair of consistent utility functions $u, v \in \mathbb{U}^a_{\mathrm{C}}$. Let $\mathcal{C}(u,v)$ be the convex enveloppe of u and v. and let $\Gamma(u)$ be the graph of $u(.)$. Three cases may occur: either

(a) $\mathcal{C}(u,v) = u(.)$, or

(b) $\mathcal{C}(u,v) = v(.)$, or

(c) $\Gamma(\mathcal{C}(u,v))$ consists in three elements: a part of $\Gamma(u)$, a part of $\Gamma(v)$ and the segment $P_u P_v$ where P_u (P_v) is the tangency point between (a) the common tangent to $\Gamma(u)$ and $\Gamma(v)$ and (b) $\Gamma(u)$ ($\Gamma(v)$). $\mathcal{C}(u,v)$ is clearly derivable, increasing and normalized.

We now claim that it is consistent (*i.e.* that $\mathcal{C}(u,v) \in \mathbb{U}^a_{\mathrm{C}}$ or, equivalently, that $epi(\mathbf{u}) \subseteq epi(\mathcal{C}(u,v))$ or, alternatively, that $hypo(\mathcal{C}(u,v)) \subseteq hypo(\mathbf{u})$) or, finally,

$$\mathbf{u}(x) \geq \mathcal{C}(u,v)(x) \; for \; all \; x \in [a,b] \tag{37}$$

This statement is clearly valid in cases (a) and (b). In the third case, let P^*_u (P^*_v) be the point of $\Gamma(\mathbf{u})$ whose abscissa is that of P_u (P_v). Then segment $P_u P_v$ must lie below segment $P^*_u P^*_v$, because, otherwise, $\mathbf{u}(.)$ could not be the superior enveloppe of the consistent utility functions. As a consequence, *the convex enveloppe* *of two consistent utility functions is a consistent utility function i.e.* it is an increasing, normalized, concave and derivable function meeting the following requirement:

$$epi(u) \cap epi(v) \supseteq epi(\mathcal{C}(u,v))) \tag{38}$$

Note that $\mathcal{C}(u,v)$ is also the convex enveloppe of $Max(u,v)$, *i.e.* we get that:

$$\mathcal{C}(u,v) = \mathcal{C}(Max(u,v))$$

The convex enveloppe of the utility functions, $\mathbf{v}$ must then coincide with their superior enveloppe $\mathbf{u}$. Indeed, for any $x \in [a,b]$, for any utility function u, we get that $\mathcal{C}(u,\mathbf{f})(x) = u(x)$ and, as a consequence, that:

$$\mathbf{u}(x) \overset{def}{=} \sup_u u(x) = \sup_u \mathcal{C}(u,\mathbf{f})(x) \leq \sup_{u,v} \mathcal{C}(u,v)(x) \overset{def}{=} \mathbf{v}(x) \tag{39}$$

and from (37) and (39) we get that:

$$\mathbf{v} = \mathbf{u}$$

Moreover, since all the $\mathcal{C}$'s are derivable so is $\mathbf{v}$. Indeed let $x,y,z \in [a,b]$ with $z < x < y$. We have:

$\mathbf{u}(y) - \mathbf{u}(x) = \mathbf{u}(y) - \mathcal{C}(u,v,w)(y) + \mathcal{C}(u,v,w)(y) - \mathcal{C}(u,v,w)(x) + \mathcal{C}(u,v,w)(x) - \mathbf{u}(x)$

$\mathbf{u}(x) - \mathbf{u}(z) = \mathbf{u}(x) - \mathcal{C}(u,v,w)(x) + \mathcal{C}(u,v,w)(x) - \mathcal{C}(u,v,w)(z) + \mathcal{C}(u,v,w)(z) - \mathbf{u}(z)$

where $\mathcal{C}(u,v,w)$ is the convex enveloppe of the consistent utility functions $u(.), v(.)$, and $w(.)$, and, consequently:

$|\frac{\mathbf{u}(y)-\mathbf{u}(x)}{y-x} - \frac{\mathcal{C}(u,v,w)(y)-\mathcal{C}(u,v,w)(x)}{y-x}| \leq |\frac{\mathbf{u}(y)-\mathcal{C}(u,v,w)(y)}{y-x}| + |\frac{\mathcal{C}(u,v,w)(x)-\mathbf{u}(x)}{y-x}|$

$|\frac{\mathbf{u}(x)-\mathbf{u}(z)}{x-z} - \frac{\mathcal{C}(u,v,w)(x)-\mathcal{C}(u,v,w)(z)}{x-z}| \leq |\frac{\mathbf{u}(x)-\mathcal{C}(u,v,w)(x)}{x-z}| + |\frac{\mathcal{C}(u,v,w)(z)-\mathbf{u}(z)}{x-z}|$

Since $\mathbf{u}(.)$ is the superior enveloppe of the utility functions, one can choose $u(.), v(.), w(.)$ such that, for any $\varepsilon > 0$

$|\mathbf{u}(x) - u(x)| \leq \varepsilon(y-x)/8$, what implies that $|\mathbf{u}(x) - \mathcal{C}(u,v,w)(x)| \leq \varepsilon(y-x)/8$

$|\mathbf{u}(y) - v(y)| \leq \varepsilon(y-x)/8$, what implies that $|\mathbf{u}(y) - \mathcal{C}(u,v,w)(y)| \leq \varepsilon(y-x)/8$

$|\mathbf{u}(z) - w(z)| \leq \varepsilon(y-x)/8$, what implies that $|\mathbf{u}(z) - \mathcal{C}(u,v,w)(z)| \leq \varepsilon(y-x)/8$

As a consequence one can choose $u(.), v(.), w(.)$ such that, for any $\varepsilon > 0$

$|\frac{\mathbf{u}(y)-\mathbf{u}(x)}{y-x} - \frac{\mathcal{C}(u,v,w)(y)-\mathcal{C}(u,v,w)(x)}{y-x}| \leq \varepsilon/8$

$|\frac{\mathbf{u}(x)-\mathbf{u}(z)}{x-z} - \frac{\mathcal{C}(u,v,w)(x)-\mathcal{C}(u,v,w)(z)}{x-z}| \leq \varepsilon/8$

Since $\mathcal{C}(u,v,w)$ is derivable we also get that, for any $\varepsilon > 0$

$|\frac{\mathcal{C}(u,v,w)(y)-\mathcal{C}(u,v,w)(x)}{y-x} - \frac{d\mathcal{C}(u,v,w)(x)}{dx}| \leq \varepsilon/8$

$|\frac{\mathcal{C}(u,v,w)(y)-\mathcal{C}(u,v,w)(x)}{y-x} - \frac{d\mathcal{C}(u,v,w)(x)}{dx}| \leq \varepsilon/8$

and, consequently:

$|\frac{\mathbf{u}(y)-\mathbf{u}(x)}{y-x} - \frac{d\mathcal{C}(u,v,w)}{dx}(x)| \leq \varepsilon/4 \;\; and \;\; |\frac{\mathbf{u}(x)-\mathbf{u}(z)}{x-z} - \frac{d\mathcal{C}(u,v,w)}{dx}(x)| \leq \varepsilon/4$

Since $\mathbf{u}(.)$ is convex, we get that

$\lim_{y\downarrow x} < \frac{\mathbf{u}(y)-\mathbf{u}(x)}{y-x} = \mathbf{u}_r^{\cdot}(x)$ and $\lim_{z\uparrow x} \frac{\mathbf{u}(x)-\mathbf{u}(z)}{x-z} = \mathbf{u}_l^{\cdot}(x)$

and, as a consequence, for any $\varepsilon > 0$:

$|\frac{\mathbf{u}(y)-\mathbf{u}(x)}{y-x} - \mathbf{u}_r^{\cdot}(x)| \leq \varepsilon/4 \;\; and \;\; |\frac{\mathbf{u}(x)-\mathbf{u}(z)}{x-z} - \mathbf{u}_l^{\cdot}(x)| \leq \varepsilon/4$

Combining the above inequalities, we get that, for any $\varepsilon > 0$

$|\mathbf{u}_r^{\cdot}(x) - \frac{d\mathcal{C}(u,v,w)}{dx}(x)| \leq \varepsilon/2 \;\; and \;\; |\mathbf{u}_l^{\cdot}(x) - \frac{d\mathcal{C}(u,v,w)}{dx}(x)| \leq \varepsilon/2$

and, consequently:

$|\mathbf{u}'_r(x) - \mathbf{u}'_l(x)| \leq \varepsilon$

Finally, $\mathbf{u}(.)$ is well derivable and its derivative is $\mathbf{u}'(x) = \mathbf{u}'_r(x) = \mathbf{u}'_l(x) = \lim\limits_{\varepsilon \to 0} \frac{d\mathcal{C}(u,v,w)}{dx}(x)$.

The last property to prove is that $\mathbf{u}(.)$ is consistent. This is well the case since we have:

$\Delta(z) \overset{def}{=} \int_a^z \mathbf{u}'(x)(F_{X_1}(x) - F_{X_2}(x))dx$

$= \int_a^z w'(x)(F_{X_1}(x) - F_{X_2}(x))dx + \int_a^z (\mathbf{u}'(x) - w'(x))(F_{X_1}(x) - F_{X_2}(x))dx$

and that the second integral may be made negligible since we have:

$|\int_a^z (\mathbf{u}'(x) - w'(x))(F_{X_1}(x) - F_{X_2}(x))dx| \leq \sup(|\mathbf{u}'(x) - w'(x)|) = \sup(|\mathbf{u}'(x) - \frac{d\mathcal{C}(u,v,w)}{dx}(x)|)$

As a consequence, the condition $\int_a^z \mathbf{u}'(x)(F_{X_1}(x) - F_{X_2}(x))dx < 0$ is equivalent to the condition $\int_a^z w'(x)(F_{X_1}(x) - F_{X_2}(x))dx < 0$.

B. We now assume that *the (standard) SSD dominance property is violated.* No concave utility functions can be consistent, *i.e.* $\mathbb{U}_{\mathrm{C}}^a = \varnothing$.[28] Finally, we get that $\mathbb{U}_{\mathrm{C}}^x = \mathbb{U}_{\mathrm{C}}$ and that *the superior enveloppe of the consistent c/c utility functions* reads:

$$\mathbf{u}(.) = \sup_{u \in \mathbb{U}_{\mathrm{C}}} (u) = \sup_{u \in \mathbb{U}_{\mathrm{C}}^x} (u)$$

Then, from standard convex analysis, we get that:

$$epi(\mathbf{u}) = \cap_{u \in \mathbb{U}_{\mathrm{C}}} epi(u)$$

and that $\mathbf{u}(.)$ *is convex*. It is also increasing and normalized. The rest of the proof is similar to the above discussion.□

9.4 Proof of Proposition 6.

We are looking for a utility function $u_{\overline{\mathbf{x}}}(.)$ over $[a, b]$, satisfying the expected utility representation, *i.e.* meeting the following requirement:[29]

$$\mathbf{E}[u_{\overline{\mathbf{x}}}(x)] = \mathcal{U}_{\overline{\mathbf{x}}}(X) \tag{40}$$

We shall successively, consider the case when X is:

(I) an arbitrary binary lottery Z belonging to any of two special subsets of $X_{\overline{\mathbf{x}}}$. whose exact definition is given below.

(II) an arbitrary binary lottery Z belonging to $X_{\overline{\mathbf{x}}}$ (*i.e.* $Z = [y, x; 1 - \pi, \pi]$ with $\pi x + (1-\pi)y = \overline{\mathbf{x}}$)

(III) an arbitrary simple lottery Z belonging to $\mathfrak{X}_{\overline{\mathbf{x}}}$

(IV) an arbitrary lottery belonging to $X_{\overline{\mathbf{x}}}$. (*i.e.* $Z \in X_{\overline{\mathbf{x}}}$)

I. The first step consists in considering two particular subsets of binary lotteries which belong to $X_{\overline{\mathbf{x}}}$ and include either the outcome a or the outcome b among their outcomes.

[28] In contrast, the subset of convex utility functions is never empty since it always includes the following function: $\underline{u}(x) = 0$ for $x \in [a, b[$ and $\underline{u}(b) = 1$.

[29] Recall that the degenerate lottery δ_z does not belong to $\mathfrak{X}_{\mathbf{x}}$ unless $z = \mathbf{x}$. Hence we must make up for this drawback and define a utility function from its expected utility representation over non degenerate lotteries..

Group A. The elements of this group are defined as indicated below:

$$\underline{X}_p^x \stackrel{def}{=} [a, x; 1-p, p]$$

where $x \in [\mathbf{\overline{x}}, b]$. Since by assumption $\underline{X}_p^x \in X_{\mathbf{\overline{x}}}$ we get that:

$$px + a(1-p) = \mathbf{\overline{x}} \tag{41}$$

or, equivalently, that:

$$p = \frac{\mathbf{\overline{x}} - a}{x - a} \tag{42}$$

Group B. The elements of this group are defined as indicated below:

$$\overline{X}_q^y \stackrel{def}{=} [y, b; q, 1-q]$$

where $y \in [a, \mathbf{\overline{x}}]$. Since, by assumption $\overline{X}_q^y \in X_{\mathbf{\overline{x}}}$ we get that:

$$qy + b\,(1-q) = \mathbf{\overline{x}} \tag{43}$$

or, equivalently, that:

$$q = \frac{b - \mathbf{\overline{x}}}{b - y} \tag{44}$$

Now recall that, by definition,

$$X_{\mathbf{\overline{x}}} \stackrel{def}{=} [a, b; 1-\pi\,(\mathbf{\overline{x}})\,, \pi\,(\mathbf{\overline{x}})\,]$$

where $x \in [\mathbf{\overline{x}}, 1]$ and

$$\pi\,(\mathbf{\overline{x}}) \stackrel{def}{=} \frac{\mathbf{\overline{x}} - a}{b - a}$$

Moreover, let $\lambda \in [0, 1]$ be defined by the following equivalence:

$$\boldsymbol{X}_{\mathbf{\overline{x}}}^{\lambda} \sim \underline{X}_p^x$$

where:

$$\boldsymbol{X}_{\mathbf{\overline{x}}}^{\lambda} \stackrel{def}{=} \lambda \delta_{\mathbf{\overline{x}}} \oplus (1-\lambda) X_{\mathbf{\overline{x}}}$$

Finally, we are looking for a utility function $u_{\mathbf{\overline{x}}}(.)$ satisfying the following expected utility representation (40):

$$pu_{\mathbf{\overline{x}}}(x) + (1-p)\,u_{\mathbf{\overline{x}}}(a) = \lambda \mathbf{\overline{x}} + (1-\lambda)\mathbf{c}(X_{\mathbf{\overline{x}}}) \tag{45}$$

where $\mathbf{c}(X_{\mathbf{\overline{x}}})$ is the certainty equivalent of $X_{\mathbf{\overline{x}}}$. Let:

$$u_{\mathbf{\overline{x}}}(x) = \frac{x - a}{\mathbf{\overline{x}} - a}[\lambda \mathbf{\overline{x}} + (1-\lambda)\,\mathbf{c}(X_{\mathbf{\overline{x}}})] - \frac{x - \mathbf{\overline{x}}}{\mathbf{\overline{x}} - a} u_{\mathbf{\overline{x}}}(a) \tag{46}$$

Then, for $x = \mathbf{\overline{x}}$, we get that $p = 1$, $\lambda = 1$, and that:

$$u_{\mathbf{\overline{x}}}(\mathbf{\overline{x}}) = \mathbf{\overline{x}} \tag{47}$$

Similarly, for $x = b$, we get that $p = \pi(\overline{\mathbf{x}})$, $\lambda = 0$, and that:

$$u_{\overline{\mathbf{x}}}(b) = \frac{b-a}{\overline{\mathbf{x}}-a}\mathbf{c}(X_{\overline{\mathbf{x}}}) - \frac{b-\overline{\mathbf{x}}}{\overline{\mathbf{x}}-a}u_{\overline{\mathbf{x}}}(a) \tag{48}$$

B. We now consider the second group of binary lotteries. From (43) we get that:

$$qy + b(1-q) = \overline{\mathbf{x}}$$

or, equivalently, that:

$$q = \frac{b-\overline{\mathbf{x}}}{b-y} \Longleftrightarrow 1-q = \frac{\overline{\mathbf{x}}-y}{b-y}$$

Next, let $\mu \in [0,1]$ be now defined by:

$$\boldsymbol{X}^{\mu}_{\overline{\mathbf{x}}} \sim \overline{X}^{y}_{q}$$

We are now looking for a utility function meeting the following requirement:

$$\mathbf{E}[u_{\overline{\mathbf{x}}}(X)] = \mathcal{U}_{\overline{\mathbf{x}}}(\boldsymbol{X}^{\mu}_{\overline{\mathbf{x}}}) \tag{49}$$

or:

$$qu_{\overline{\mathbf{x}}}(y) + (1-q)\,u_{\overline{\mathbf{x}}}(b) = \mu x + (1-\mu)\mathbf{c}(X_{\overline{\mathbf{x}}}) \tag{50}$$

or, equivalently:

$$\frac{b-\overline{\mathbf{x}}}{b-y}u_{\overline{\mathbf{x}}}(y) + \frac{\overline{\mathbf{x}}-y}{b-y}u_{\overline{\mathbf{x}}}(b) = \mu x + (1-\mu)\mathbf{c}(X_{\overline{\mathbf{x}}})$$

or, finally:

$$u_{\overline{\mathbf{x}}}(y) = \frac{b-y}{b-\overline{\mathbf{x}}}[\mu\overline{\mathbf{x}} + (1-\mu)\mathbf{c}(X_{\overline{\mathbf{x}}})] - \frac{\overline{\mathbf{x}}-y}{b-\overline{\mathbf{x}}}u_{\overline{\mathbf{x}}}(b) \tag{51}$$

Hence for $y = x$, we get that $q = 1$, $\mu = 1$, and, again, equation (47). For $y = a$, we get that $q = 1 - \pi(\overline{\mathbf{x}})$, $\mu = 0$, and:

$$u_{\overline{\mathbf{x}}}(a) = \frac{b-a}{b-\overline{\mathbf{x}}}\mathbf{c}(X_{\overline{\mathbf{x}}}) - \frac{\overline{\mathbf{x}}-a}{b-\overline{\mathbf{x}}}u_{\overline{\mathbf{x}}}(b) \tag{52}$$

Clearly equations (48) and (52) are the same and can be rewritten as:

$$\mathbf{E}[u_{\overline{\mathbf{x}}}(X_{\overline{\mathbf{x}}})] = \frac{\overline{\mathbf{x}}-a}{b-a}u_{\overline{\mathbf{x}}}(b) + \frac{b-\overline{\mathbf{x}}}{b-a}u_{\overline{\mathbf{x}}}(a) = \mathbf{c}(X_{\overline{\mathbf{x}}}) \tag{53}$$

The normalization conditions of the utility function come down to (47) and (53). Finally, note that we have the following relations:

$$\begin{aligned}
\mathcal{U}_{\overline{\mathbf{x}}}\left(\underline{X}^{x}_{p}\right) &= \mathbf{E}[u_{\overline{\mathbf{x}}}(\underline{X}^{x}_{p})] = \frac{\overline{\mathbf{x}}-a}{x-a}u_{\overline{\mathbf{x}}}(x) + \frac{x-\overline{\mathbf{x}}}{x-a}u_{\overline{\mathbf{x}}}(a) \\
\mathcal{U}_{\overline{\mathbf{x}}}\left(\overline{X}^{y}_{q}\right) &= \mathbf{E}[u_{\overline{\mathbf{x}}}(\overline{X}^{y}_{q})] = \frac{\overline{\mathbf{x}}-y}{b-y}u_{\overline{\mathbf{x}}}(b) + \frac{b-\overline{\mathbf{x}}}{b-y}u_{\overline{\mathbf{x}}}(x) \\
\mathcal{U}_{\overline{\mathbf{x}}}(X_{\overline{\mathbf{x}}}) &= \mathbf{E}[u_{\overline{\mathbf{x}}}(X_{\overline{\mathbf{x}}})] = \mathbf{c}(X_{\overline{\mathbf{x}}}) \\
\mathcal{U}_{\overline{\mathbf{x}}}(\delta_{\overline{\mathbf{x}}}) &= \mathbf{E}[u_{\overline{\mathbf{x}}}(\delta_{\overline{\mathbf{x}}})] = u_{\overline{\mathbf{x}}}(\overline{\mathbf{x}}) = \overline{\mathbf{x}}
\end{aligned}$$

$$u_{\overline{\mathbf{x}}}\left(b\right)=\frac{b-a}{\overline{\mathbf{x}}-a}\mathbf{c}(X_{\overline{\mathbf{x}}})-\frac{b-\overline{\mathbf{x}}}{\overline{\mathbf{x}}-a}u_{\overline{\mathbf{x}}}(a)$$

II. We now show that the above result holds *for all binary lotteries* belonging to $X_{\overline{\mathbf{x}}}$. Indeed, let $Z=[y,x;1-\pi,\pi]$ and assume that $Z\in X_{\overline{\mathbf{x}}}$. As a consequence, we get that:

$$\pi x+\left(1-\pi\right)y=\overline{\mathbf{x}} \tag{54}$$

Consider the two following compound lotteries:

$$\alpha\underline{X}_p^x\oplus\left(1-\alpha\right)\overline{X}_q^y$$

and

$$\beta Z\oplus\left(1-\beta\right)X_{\overline{\mathbf{x}}}$$

where $\alpha,\beta,\in[0,1]$. The two lotteries have the same support $\{0,x,y,1\}$ and will coincide[30] iff they exhibit the same probabilities, *i.e. iff*:

$$\begin{aligned}\alpha(1-p)&=(1-\beta)(1-\pi(\overline{\mathbf{x}}))\\(1-\alpha)q&=(1-\pi)\beta\\\alpha p&=\pi\beta\\(1-\alpha)(1-q)&=\pi(\overline{\mathbf{x}})(1-\beta)\end{aligned} \tag{55}$$

Actually there are only two independent equations among the four above because

(a) the probabilities sum to one –we may therefore leave aside the last equation– and since

(b) p and q both depend on x.[31] Indeed, combining the three remaining equations gives an equation which is nothing but (54).

From the linearity of $\mathcal{U}_{\overline{\mathbf{x}}}(.)$ we also get that:

$$\begin{aligned}\mathcal{U}_{\overline{\mathbf{x}}}(\alpha\underline{X}_p^x\oplus(1-\alpha)\overline{X}_q^y)&=\alpha\mathcal{U}_{\overline{\mathbf{x}}}(\underline{X}_p^x)+(1-\alpha)\mathcal{U}_{\overline{\mathbf{x}}}(\overline{X}_q^y)\\\mathcal{U}_{\overline{\mathbf{x}}}\left(\beta Z\oplus(1-\beta)X_{\overline{\mathbf{x}}}\right)&=\beta\mathcal{U}_{\overline{\mathbf{x}}}\left(Z\right)+(1-\beta)\mathcal{U}_{\overline{\mathbf{x}}}\left(X_{\overline{\mathbf{x}}}\right)\end{aligned}$$

and, consequently:

$$\begin{aligned}\mathcal{U}_{\overline{\mathbf{x}}}\left(Z\right)&=\beta^{-1}\left[\alpha\mathcal{U}_{\overline{\mathbf{x}}}(\underline{X}_p^x)+(1-\alpha)\mathcal{U}_{\overline{\mathbf{x}}}(\overline{X}_q^y)-(1-\beta)\mathcal{U}_{\overline{\mathbf{x}}}\left(X_{\overline{\mathbf{x}}}\right)\right]\\&=\frac{\alpha}{\beta}\mathcal{U}_{\overline{\mathbf{x}}}(\underline{X}_p^x)+\frac{(1-\alpha)}{\beta}\mathcal{U}_{\overline{\mathbf{x}}}(\overline{X}_q^y)-\frac{(1-\beta)}{\beta}\mathcal{U}_{\overline{\mathbf{x}}}\left(X_{\overline{\mathbf{x}}}\right)\end{aligned}$$

and, substituting α and β for their values in (55) we get that:

$$\mathcal{U}_{\overline{\mathbf{x}}}(Z)=\frac{\pi}{p}\mathcal{U}_{\overline{\mathbf{x}}}(\underline{X}_p^x)+\frac{1-\pi}{q}\mathcal{U}_{\overline{\mathbf{x}}}(\overline{X}_q^y)-\frac{(1-q)(1-\pi)}{q\pi(\overline{\mathbf{x}})}\mathcal{U}_{\overline{\mathbf{x}}}\left(X_{\overline{\mathbf{x}}}\right)$$

[30] *i.e.* $\alpha\underline{X}_p^d\oplus(1-\alpha)\overline{X}_q^c=\beta Z\oplus(1-\beta)X_{\mathbf{x}}$

[31] They are linked together by the following formula: $q=d\left(1-p\right)/\left(d-c\right)$

or equivalently:

$$\begin{aligned}\mathcal{U}_{\overline{\mathbf{x}}}(Z)=&\pi u_{\overline{\mathbf{x}}}(x)+(1-\pi)u_{\overline{\mathbf{x}}}(y)+\pi\frac{x-\overline{\mathbf{x}}}{x-a}u_{\overline{\mathbf{x}}}(a)\\&+(1-\pi)\frac{\overline{\mathbf{x}}-y}{b-\overline{\mathbf{x}}}u_{\overline{\mathbf{x}}}(b)-\frac{(1-q)(1-\pi)}{q\pi(\overline{\mathbf{x}})}u_{\overline{\mathbf{x}}}(\mathbf{c}(X_{\overline{\mathbf{x}}}))\end{aligned}$$

and, finally:

$$\begin{aligned}\mathcal{U}_{\overline{\mathbf{x}}}(Z)=&\,\mathbf{E}[u_{\overline{\mathbf{x}}}(Z)]+\frac{\overline{\mathbf{x}}-y}{x-y}\frac{x-\overline{\mathbf{x}}}{b-\overline{\mathbf{x}}}u_{\overline{\mathbf{x}}}(a)+\frac{x-\overline{\mathbf{x}}}{b-\overline{\mathbf{x}}}\frac{\overline{\mathbf{x}}-y}{b-\overline{\mathbf{x}}}u_{\overline{\mathbf{x}}}(b)\\&-\frac{\overline{\mathbf{x}}-y}{b-\overline{\mathbf{x}}}\frac{x-\overline{\mathbf{x}}}{x-y}\frac{b-a}{\overline{\mathbf{x}}-a}[\pi(\overline{\mathbf{x}})u_{\overline{\mathbf{x}}}(b)+(1-\pi(\overline{\mathbf{x}}))u_{\overline{\mathbf{x}}}(a)]\\&=\mathbf{E}[u_{\overline{\mathbf{x}}}(Z)]+\frac{\overline{\mathbf{x}}-y}{x-y}\frac{x-\overline{\mathbf{x}}}{b-\overline{\mathbf{x}}}u_{\overline{\mathbf{x}}}(a)+\frac{x-\overline{\mathbf{x}}}{b-\overline{\mathbf{x}}}\frac{\overline{\mathbf{x}}-y}{b-\overline{\mathbf{x}}}u_{\overline{\mathbf{x}}}(b)\\&-\left[\frac{\overline{\mathbf{x}}-y}{b-\overline{\mathbf{x}}}\frac{x-\overline{\mathbf{x}}}{x-y}u_{\overline{\mathbf{x}}}(b)+\frac{\overline{\mathbf{x}}-y}{x-y}\frac{x-\overline{\mathbf{x}}}{\overline{\mathbf{x}}-a}u_{\overline{\mathbf{x}}}(a)\right]\\&=\mathbf{E}[u_{\overline{\mathbf{x}}}(Z)]\end{aligned}$$

III. Next, from a straightforward induction method, we can derive, for any simple lottery whose expected value is $\overline{\mathbf{x}}$:

$$\mathbf{E}[u_{\overline{\mathbf{x}}}(Z)]=\sum p_Z(x)u_{\overline{\mathbf{x}}}(x))=\mathcal{U}_{\overline{\mathbf{x}}}(Z)$$

Thus, the utility of a simple lottery equals the expected utility of its consequences. However, the induction argument is valid only if the number of possible consequences is finite. Hence, It remains to be shown that Axiom 3 implies expected utility maximization over the whole subset $\mathfrak{X}_{\overline{\mathbf{x}}}$.

IV. To obtain the expected utility representation over the entire subset,[32] one can proceed as indicated below. Any lottery $X\in\mathfrak{X}_{\overline{\mathbf{x}}}$ whose *c.d.f.* $F_X(.)$ is continuous can be viewed as the limit of two sequences of simple lotteries whose expected value is x and which either SSSD dominate X or are SSSD dominated by X. To see this, recall that we have:

$$\delta_{\overline{\mathbf{x}}}\succsim X\succsim X_{\overline{\mathbf{x}}}$$

and consider the two sequences of simple lotteries $\{X_n^*\}_{n\in\mathbb{N}}$ and $\{X_n^{**}\}_{n\in\mathbb{N}}$ where X_n^* and X_n^{**} both belong to $\mathfrak{X}_{\overline{\mathbf{x}}}$ and which are defined as indicated on Figure 3.

Step 1.
Lotteries X_1^* and X_1^{**} are such that:
$F_{X_1^*}(x)=F_X\left(x_1^2\right)$ if $x\in[x_1^1,x_1^3[$; $F_{X_1^*}(x)=1$ if $x=x_1^3$
$F_{X_1^{**}}(x)=0$ if $x\in[x_1^1,x_1^2[$; $F_{X_1^{**}}(x)=1$ if $x\in[x_1^2,x_1^3[$, where:
$x_1^1{=}a$; $x_1^2{=}\,\mathbf{E}[X]=\overline{\mathbf{x}}$; $x_1^3{=}\,b$ and:

[32] Note that we do not need any additional axiom (dominance and/or monotonicity) since these further axioms are are implied by the property of strict risk aversion.

$\mathbf{E}[X] = \int_a^b x dF_X(x) \Longleftrightarrow F_X(\mathbf{E}[X]) = \frac{b-\mathbf{E}[X]}{b-a}$

Step 2.

Lotteries X_2^* and X_2^{**} are defined as indicated below:

$F_{X_2^*}(x) = F_X(x_2^2)$ if $x \in [x_2^1, x_2^3[$; $F_{X_2^*}(x) = F_X(x_2^4)$ if $x \in [x_2^3, x_2^5[$; $F_{X_2^*}(x) = 1$ if $x = x_2^5$

$F_{X_2^{**}}(x) = 0$ if $x \in [x_2^1, x_2^2[$; $F_{X_2^{**}}(x) = F_X(x_2^3)$ if $x \in [x_2^2, x_2^4[$; $F_{X_2^{**}}(x) = 1$ if $x \in [x_2^4, x_2^5]$

where:

$x_2^1 \overset{def}{=} x_1^1 = a$; $x_2^3 \overset{def}{=} x_1^2 = \mathbf{E}[X] = \overline{\mathbf{x}}$; $x_2^5 \overset{def}{=} x_1^3 = b$

and where x_2^2 and x_2^4 are defined by the following conditions:

$\int_{x_1^1}^{x_1^2} x dF_X(x) = x_2^2 F_X(x_2^3) = x_2^2(F_X(x_1^2) - F_X(x_1^1))$

$\int_{x_1^2}^{x_1^3} x dF_X(x) = x_2^4(1 - F_X(x_1^2)) = x_2^4 F_X(x_1^3) - F_X(x_1^2))$

....

Step n. Finally, $\{X_n^*\}_{n\in\mathbb{N}}$ and $\{X_n^{**}\}_{n\in\mathbb{N}}$ are defined recursively by the following equations:

$$x_{n+1}^{2i-1} = x_n^i \qquad \text{if } i = 1, 2^n+1 \tag{56}$$

$$x_{n+1}^{2i} = \left(\int_{x_n^i}^{x_n^{i+1}} x dF_X(x)\right) / (F_X(x_n^{i+1}) - F_X(x_n^i)) \quad \text{if } i = 1, 2^n \tag{57}$$

$$F_{X_{n+1}^*}(x) = F_X(x_{n+1}^{2i}) \quad \text{if } x \in [x_{n+1}^{2i-1}, x_{n+1}^{2i+1}[\text{ and } i = 1, 2^n \tag{58}$$

$$F_{X_{n+1}^{**}}(x) = F_X(x_{n+1}^{2i+1}) \quad \text{if } x \in [x_{n+1}^{2i}, x_{n+1}^{2i+2}[\text{ and } i = 1, 2^n \tag{59}$$

given that we have:

$F_{X_{n+1}^*}(x_{n+1}^{2^n+1}) = F_{X_{n+1}^{**}}(x_{n+1}^{2^n+1}) = 1$ and:

$F_{X_{n+1}^{**}}(x) = F_X(x_n^1) = 0$ if $x \in [x_{n+1}^1, x_{n+1}^2[$.

Whatever the value of x, the two sequences of simple lotteries which we have built converge towards X. To see this, first recall that the sequence whose general term is $\mid F_{X_n^{**}}(x) - F_{X_n^*}(x) \mid$ converges towards a limit $\ell_x \geq 0$ since it is a decreasing sequence of positive real numbers. Hence, for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that:

$$n \geq N \Rightarrow \ell_x \leq \mid F_{X_n^*}(x) - F_{X_n^{**}}(x) \mid \leq \ell_x + \varepsilon \tag{60}$$

Actually ℓ_x must be zero otherwise a contradiction would appear. To see this, assume that $\ell_x > 0$ and consider the subdivision $\{x_N^i\}_{i=1,2^{N-1}+1}$. Assume provisionally that $x \in [x_N^{2k}, x_{N-1}^{k+1}[$. From (58) and (59) we get that:

$$F_{X_N^*}(x) = F_X(x_N^{2k}) \textit{ and } F_{X_N^{**}}(x) = F_X(x_N^{2k+1}) = F_X(x_{N-1}^{k+1})$$

From (60) we get that:

$$F_{X_N^*}(x) - F_{X_N^{**}}(x) = F_X(x_N^{2k}) - F_X(x_N^{2k+1}) \geq \ell_x \tag{61}$$

Consider the next subdivision $\{x_{N+1}^{i}\}_{i=1,(2)^{N+1}+1}$ and assume, again provisionally, that $x \in [x_N^{2k}, x_{N+1}^{4k}[$. From (58) and (59) we get that:

$$F_{X^*_{N+1}}(x) = F_X(x_{N+1}^{4k}) \text{ and } F_{X^{**}_{N+1}}(x) = F_X(x_{N+1}^{4k+1}) = F_X(x_N^{2k})$$

From (60) we get that:

$$F_{X^*_{N+1}}(x) - F_{X^{**}_{N+1}}(x) = F_X(x_{N+1}^{4k+1}) - F_X(x_N^{2k}) \geq \ell_x$$

Let $y \in [x_{N+1}^{4k}, x_N^{2k+1}[$. From (58) and (59) we get that:

$$F_{X^*_{N+1}}(y) = F_X(x_{N+1}^{4k}) \; and \; F_{X^{**}_{N+1}}(y) = F_X(x_{N+1}^{4k+1}) = F_X(x_N^{2k+1})$$

From (60) we get that:

$$F_{X^*_{N+1}}(y) - F_{X^{**}_{N+1}}(y) = F_X(x_{N+1}^{4k}) - F_X(x_N^{2k+1}) \geq \ell_y$$

Now, since we have $F_X(x_N^{2k}) - F_X(x_{N+1}^{4k}) \geq 0$ and $F_X(x_{N+1}^{4k}) - F_X(x_N^{2k+1}) \geq 0$ and using (61) we get that:

$$\begin{aligned} &| \quad F_X(x) - F_{X_n^*}(x) \mid =\mid F_X(x_N^{2k}) - F_X(x_N^{2k+1}) \mid \\ = \quad &\mid F_X(x_N^{2k}) - F_X(x_{N+1}^{4k}) \mid + \mid F_X(x_{N+1}^{4k}) - F_X(x_N^{2k+1}) \mid \geq \ell_x + \ell_y \end{aligned}$$

As a consequence we cannot have $\left|F_X(x) - F_{X_n^*}(x)\right| \in [\ell_x, \ell_x + \varepsilon]$ for any infinitesimal quantity ε unless $\ell_y = 0$. Such a conclusion clearly does not depend on the initial choice of x and y. Finally, the two sequences both converge towards X *i.e.* :

$$\lim_{n\to\infty} X_n^{**} = \lim_{n\to\infty} X_n^* = X \tag{62}$$

Since $U_{\overline{\mathbf{x}}}(.)$ is continuous we also get that:

$$\lim_{n\to\infty} \mathcal{U}_{\overline{\mathbf{x}}}(X_n^{**}) = \mathcal{U}_{\overline{\mathbf{x}}}(X) = \lim_{n\to\infty} \mathcal{U}_{\overline{\mathbf{x}}}(X_n^*) \tag{63}$$

Now recall that, by definition, the two sequences of simple lotteries which we have built are such that:
(a) X_n^* (X_n^{**}) is a mean preserving spread (henceforth MPS) of X_{n-1}^* (X_{n+1}^{**}),
(b) X_n^* and X_n^{**} belong to $\mathfrak{X}_{\overline{\mathbf{x}}}$
(c) X_n^* is a MPS of X which, in its turn, is a MPS of X_n^{**}.

Hence, we get that:

$\delta_{\overline{\mathbf{x}}} \succsim X_1^{**} \succsim ... \succsim X_n^{**} \succsim X \succsim X_n^* \succsim ... \succsim X_1^* \succsim X_{\overline{\mathbf{x}}}$

or, equivalently:

$\mathcal{U}_{\overline{\mathbf{x}}}(\delta_{\overline{\mathbf{x}}}) \geq \mathcal{U}_{\overline{\mathbf{x}}}(X_1^{**}) \geq ... \geq \mathcal{U}_{\overline{\mathbf{x}}}(X_n^{**}) \geq \mathcal{U}_{\overline{\mathbf{x}}}(X) \geq \mathcal{U}_{\overline{\mathbf{x}}}(X_n^*) \geq ... \geq \mathcal{U}_{\overline{\mathbf{x}}}(X_1^*) \geq \mathcal{U}_{\overline{\mathbf{x}}}(X_{\overline{\mathbf{x}}})$

or, alternatively:

$\mathcal{U}_{\overline{\mathbf{x}}}(\delta_{\overline{\mathbf{x}}}) \geq \mathbf{E}[u_{\overline{\mathbf{x}}}(X_1^{**})] \geq ... \geq \mathbf{E}[u_{\overline{\mathbf{x}}}(X_n^{**})] \geq ...\mathcal{U}_{\overline{\mathbf{x}}}(X)... \geq \mathbf{E}[u_{\overline{\mathbf{x}}}(X_n^*)] \geq ... \geq \mathbf{E}[u_{\overline{\mathbf{x}}}(X_1^*)] \geq \mathcal{U}_{\overline{\mathbf{x}}}(X_{\overline{\mathbf{x}}})$

and, finally:

$$\lim_{n\to\infty} \mathbf{E}[u_{\overline{\mathbf{x}}}(X_n^{**})] = \mathcal{U}_{\overline{\mathbf{x}}}(X) = \lim_{n\to\infty} \mathbf{E}[u_{\overline{\mathbf{x}}}(X_n^*)] \tag{64}$$

Now we have to show that $\mathcal{U}_{\overline{\mathbf{x}}}(X) = \mathbf{E}[u_{\overline{\mathbf{x}}}(X)]$. To do so, we just do as before, given that the subdivisions are now defined as indicated below:

$$u_{n+1}^{2i-1} = u_n^i$$
$$\int_{u_n^i}^{u_n^{i+1}} u dG_U(u) = u_n^{2i}((G_U(u_n^{i+1}) - G_U(u_n^i)) \quad \text{for } i = 1, 2^n$$

where $u = u_{\overline{\mathbf{x}}}(x)$ $u_n^i = u_{\overline{\mathbf{x}}}(x_n^i)$, $G_U(u) = G_U(u_{\overline{\mathbf{x}}}(x)) = F_X(x)$
and $\int_0^1 u dG_U(u) = \int_a^b u_{\overline{\mathbf{x}}}(x) dF_X(x)$.

$$G_{U_{n+1}^*}(u) = G_U(u_{n+1}^{2i}) \quad \text{for } u \in \left[u_{n+1}^{2i-1}, u_{n+1}^{2i+1}\right[\text{ and } i = 1, 2^n$$
$$G_{U_{n+1}^{**}}(u) = G_U(u_{n+1}^{2i+1}) \quad \text{for } u \in \left[u_{n+1}^{2i}, u_{n+1}^{2i+2}\right[\text{ and } i = 1, 2^n$$

Finally, the two sequences $\{\mathbf{E}[u_{\overline{\mathbf{x}}}(X_n^*)]\}_{n\in\mathbb{N}}$ and $\{\mathbf{E}[u_{\overline{\mathbf{x}}}(X_n^{**})]\}_{n\in\mathbb{N}}$ have the same limit $\mathcal{U}_{\overline{\mathbf{x}}}(X)$ which is but $\mathbf{E}[u_{\overline{\mathbf{x}}}(X)]$.□

9.5 Proof of Proposition 8.

Let $g(z, z-x) \overset{def}{=} u_z(x) - z$ and $z \overset{def}{=} \mathbf{E}[X]$. Invariance by translation then implies that:

$$z + \Delta x - \int_a^b (g(z+\Delta x, z-x) + z + \Delta x) dF_X(x) = z - \int_a^b (g(z, z-x) + z) dF_X(x)$$

or:

$$\int_a^b g(z + \Delta x, z - x) dF_X(x) = \int_a^b g(z, z-x) dF_X(x)$$

Since the above equality is valid for any *c.d.f.* $F_X(.)$ and any values of x, z and Δx, we get that:

$$g(z + \Delta x, z - x) - g(z, z - x) = 0$$

i.e. that $g(z, z-x)$ depends only on $z - x$. Hence we set:

$$g(z, z - x) = \mathcal{E}(x - z)$$

where $\mathcal{E}(.)$ is an arbitrary function. Finally we get: $u_z(z) = z + \mathcal{E}(x - z)$. $\mathcal{E}(.)$ is continuous and increasing because, from Proposition 6, $u_z(.)$ is continuous and increasing. To see this, recall that $u_0(x) = \mathcal{E}(x)$. □

9.6 Proof of Proposition 13.

The first part of the proof consists in proving that, in LS-models, two lotteries X_1 and X_2 which have the same expected utility $\overline{\mathbf{u}}$ and the same certainty equivalent $\mathbf{c}$, are strongly equivalent. Let X_1 and X_2 exhibit the same expected utility $\overline{\mathbf{u}}$ and the same certainty equivalent $\mathbf{c}$. From (22) we get, for $i = 1, 2$:

$$\mathbf{u}(\mathbf{c}) = \overline{\mathbf{u}} + \sum_{n=1}^N p_n^i \mathcal{E}[(\mathbf{u}(x_n) - \overline{\mathbf{u}})]$$

where $X_i = \left[x_1, ..., x_N \; ; \; p_1^i, ..., p_N^i\right]$ $(i = 1, 2)$ and where $\overline{\mathbf{u}} = \sum_{n=1}^N p_n^i \mathbf{u}(x_n)$. As a consequence, we have:

$$\sum_{n=1}^N p_n^1 \mathcal{E}[(\mathbf{u}(x_n) - \overline{\mathbf{u}})] - \sum_{n=1}^N p_n^2 \mathcal{E}[(\mathbf{u}(x_n) - \overline{\mathbf{u}})] = 0 \tag{65}$$

Now, consider the compound lottery

$$X_\lambda \stackrel{def}{=} \lambda X_1 \oplus (1-\lambda) X_2 = \left[x_1, ..., x_N \; ; \; \lambda p_1^1 + (1-\lambda) p_1^2, ..., \lambda p_N^1 + (1-\lambda) p_N^2\right]$$

Its expected utility is:

$$\mathbf{E}\left[\mathbf{u}(X_\lambda)\right] = \sum_{n=1}^N \left(\lambda p_n^1 + (1-\lambda) p_n^2\right) \mathbf{u}\left(x_n\right) = \overline{\mathbf{u}}$$

From (22) we also get that:

$$\mathbf{u}(\mathbf{c}(X_\lambda)) = \overline{\mathbf{u}} + \sum_{n=1}^N \left(\lambda p_n^1 + (1-\lambda) p_n^2\right) \left(\mathcal{E}[\mathbf{u}(x_n) - \overline{\mathbf{u}}]\right)$$

where $\mathbf{c}(X_\lambda)$ is the certainty equivalent of X_λ and, finally:

$$\mathbf{u}(\mathbf{c}(X_\lambda)) - \mathbf{u}\left(\mathbf{c}\right) = \lambda\left(\sum_{n=1}^N p_n^1 \mathcal{E}[(\mathbf{u}(x_n) - \overline{\mathbf{u}})] - \sum_{n=1}^N p_n^2 \mathcal{E}[(\mathbf{u}(x_n) - \overline{\mathbf{u}})]\right) = 0$$

The proof of the converse is as follows. We must show that if X_1 and X_2 are strongly equivalent –i.e. if they have the same certainty equivalent and if they exhibit the betweenness property–, then they exhibit the same expected utility. To do so, we consider two discrete lotteries:

$$X_i = \left[x_1, ..., x_N \; ; \; p_1^i, ..., p_N^i\right] \quad i = 1, 2$$

and their $(\lambda, 1-\lambda)$-mixing:

$$\lambda X_1 \oplus (1-\lambda) X_2 = \left[x_1, ..., x_N \; ; \; \lambda p_1^1 + (1-\lambda) p_1^2, ..., \lambda p_N^1 + (1-\lambda) p_N^2\right]$$

where $\lambda \in [0, 1]$.

We assume that they have the same certainty equivalent. Hence, we have, for $i = 1, 2$:

$$\mathbf{u}\left(c\right) = \mathbf{u}\left(\mathbf{c}\left(X_i\right)\right) = \overline{\mathbf{u}}_i + \sum_{n=1}^N p_n^i \mathcal{E}\left(\mathbf{u}_n^i\right) \tag{66}$$

where:

$$\overline{\mathbf{u}}_i = \sum_{n=1}^N p_n^i \mathbf{u}(x_n) \text{ and } \mathbf{u}_n^i = \mathbf{u}(x_n) - \overline{\mathbf{u}}_i \tag{67}$$

Now, recall that, by definition, we have:

$$\begin{aligned} \mathbf{u}(\mathbf{c}(\lambda X_1 \oplus (1-\lambda) X_2)) = {} & \lambda \overline{\mathbf{u}}_1 + (1-\lambda) \overline{\mathbf{u}}_2 \\ & + \sum_{n=1}^N \left(\lambda p_n^1 + (1-\lambda) p_n^2\right) \left(\mathcal{E}[\lambda \mathbf{u}_n^1 + (1-\lambda) \mathbf{u}_n^2]\right) \end{aligned}$$

and, from (66) and (67), we get that:

$$\lambda\mathbf{u}(\mathbf{c}(X_1)) + (1-\lambda)\mathbf{u}(\mathbf{c}(X_2)) = \lambda\overline{\mathbf{u}}_1 + (1-\lambda)\overline{\mathbf{u}}_2 + \sum_{n=1}^{N} \lambda p_n^1 \mathcal{E}\left(\mathbf{u}_n^1\right) + \sum_{n=1}^{N} (1-\lambda) p_n^2 \mathcal{E}\left(\mathbf{u}_n^2\right)$$

Now, from the betweenness property we get that:
$u(\mathbf{c}(\lambda X_1 \oplus (1-\lambda) X_2) = \lambda u\left(\mathbf{c}\left(X_1\right)\right) + (1-\lambda) u\left(\mathbf{c}\left(X_2\right)\right)$,
and, consequently:

$$\begin{array}{l} \sum_{n=1}^{N} \lambda p_n^1 \mathcal{E}[\mathbf{u}_n^1] \\ +(1-\lambda) p_n^2 \mathcal{E}[\mathbf{u}_n^2] \end{array} = \lambda\left(\sum_{n=1}^{N} p_n^1 \mathcal{E}[\lambda\mathbf{u}_n^1 + (1-\lambda)\mathbf{u}_n^2]\right) + (1-\lambda)\left(\sum_{n=1}^{N} p_n^2 \mathcal{E}[\lambda\mathbf{u}_n^1 + (1-\lambda)\mathbf{u}_n^2]\right)$$

or, equivalently:

$$\begin{array}{l} \sum_{n=1}^{N} \lambda p_n^1 \mathcal{E}[\mathbf{u}_n^1] + (1-\lambda) p_n^2 \mathcal{E}[\mathbf{u}_n^1] \\ +p_n^2 (1-\lambda)\left(\mathcal{E}[\mathbf{u}_n^2] - \mathcal{E}\left(\mathbf{u}_n^1\right)\right) \end{array} = \sum_{n=1}^{N} \varpi_n(\lambda) \mathcal{E}[\lambda\mathbf{u}_n^1 + (1-\lambda)\mathbf{u}_n^2]$$

where $\varpi_n(\lambda) \stackrel{def}{=} (\lambda p_n^1 + (1-\lambda) p_n^2)$. Hence, we get the following equality:

$$\sum_{n=1}^{N} \varpi_n(\lambda)(\mathcal{E}[\mathbf{u}_n^1] - \mathcal{E}[\lambda\mathbf{u}_n^1 + (1-\lambda)\mathbf{u}_n^2]) = \sum_{n=1}^{N} p_n^2 (1-\lambda)(\mathcal{E}\left(\mathbf{u}_n^1\right) - \mathcal{E}\left(\mathbf{u}_n^2\right))$$

or, equivalently:

$$\sum_{n=1}^{N} \varpi_n(\lambda) \left[\begin{array}{c} \mathcal{E}[\mathbf{u}(x_n) - \overline{\mathbf{u}}_1] \\ -\mathcal{E}[\mathbf{u}(x_n) - \overline{\mathbf{u}}_\lambda] \end{array} \right] (1-\lambda)^{-1} = \sum_{n=1}^{N} p_n^2 \left(\begin{array}{c} \mathcal{E}[\mathbf{u}(x_n) - \overline{\mathbf{u}}_1] \\ -\mathcal{E}[\mathbf{u}(x_n) - \overline{\mathbf{u}}_2] \end{array} \right)$$

where $\overline{\mathbf{u}}_\lambda = \lambda\overline{\mathbf{u}}_1 + (1-\lambda)\overline{\mathbf{u}}_2$. Finally we get the following equality:

$$\begin{aligned} & \sum_{n=1}^{N} \varpi_n(\lambda) \left(\overline{\mathbf{u}}_2 - \overline{\mathbf{u}}_1\right) \mathcal{E}'\left(\mathbf{u}(x_n) - \overline{\mathbf{u}}_1 + \theta_n^\lambda (1-\lambda)\left(\overline{\mathbf{u}}_2 - \overline{\mathbf{u}}_1\right)\right) \\ = \; & \sum_{n=1}^{N} p_n^2 \left(\overline{\mathbf{u}}_2 - \overline{\mathbf{u}}_1\right) \mathcal{E}'\left(\mathbf{u}(x_n) - \overline{\mathbf{u}}_1 + \zeta_n^\lambda (\overline{\mathbf{u}}_2 - \overline{\mathbf{u}}_1)\right) \end{aligned}$$

which may be rewritten as:

$$\left(\overline{\mathbf{u}}_2 - \overline{\mathbf{u}}_1\right) F(\lambda) = \left(\overline{\mathbf{u}}_2 - \overline{\mathbf{u}}_1\right) \Phi(\lambda)$$

Since $F(\lambda)$ cannot be equal to $\Phi(\lambda)$ for any value of λ, we must have $\overline{\mathbf{u}}_1 - \overline{\mathbf{u}}_2 = 0$.□

9.7 Proof of Proposition 14.

The lotteries $\underline{X}_p^x$ and X will be strongly indifferent *iff*:

$$\begin{array}{r} p\mathbf{u}(x) = \pi \\ p\mathbf{u}(x) + p\mathcal{E}\left(\mathbf{u}(x) - p\mathbf{u}(x)\right) + (1-p)\,\mathcal{E}\left(-p\mathbf{u}(x)\right) = \mathbf{c}(X) \end{array}$$

where:

$$\pi \overset{def}{=} \mathbf{u}\left(\mathbf{z}\left(X\right)\right) = \mathbf{E}\left[\mathbf{u}\left(X\right)\right] \in [0,1]$$

$$\mathbf{c}(X) \overset{def}{=} \mathbf{u}\left(\mathbf{c}\left(X\right)\right) = \pi + \mathbf{E}\left[\mathcal{E}\left(\mathbf{u}\left(X\right) - \pi\right)\right] \in \left[\mathbf{u}_{\pi}, \pi\right]$$

$$\mathbf{u}_{\pi} = \pi + \pi\mathcal{E}\left(1-\pi\right) + \left(1-\pi\right)\mathcal{E}\left(-\pi\mathbf{u}(x)\right)$$

The above system can be rewritten as indicated below:

$$\mathbf{u}(x) = \pi/p$$

$$\pi + p\mathcal{E}\left((\pi/p) - \pi\right) + (1-p)\,\mathcal{E}\left(-\pi\right) = \mathbf{c}(X)$$

the first equation is checked *iff* $x = u^{-1}(\pi/p)$. The second equation has a unique solution because the function

$$\varphi(p) = \pi + p\mathcal{E}\left((\pi/p) - \pi\right) + (1-p)\mathcal{E}\left(-\pi\right)$$

is increasing and maps$[\pi, 1]$ over $[\mathbf{u}_{\pi}, \pi]$. Indeed we have:

$$\begin{aligned} \varphi'(p) &= \mathcal{E}\left(((\pi/p) - \pi\right) - \mathcal{E}\left(-\pi\right) - \pi/p\mathcal{E}'\left((\pi/p) - \pi\right) \\ &= \mathcal{E}\left(-\pi\right) + (\pi/p)\mathcal{E}'(\theta(\pi/p) - \pi) - \mathcal{E}\left(-\pi\right) - \pi/p\mathcal{E}'\left((\pi/p) - \pi\right) \\ &= (\pi/p)\left[\mathcal{E}'\left(\theta(\pi/p) - \pi\right) - \mathcal{E}'\left((\pi/p) - \pi\right)\right] \end{aligned}$$

and $\varphi'(p)$ is well negative since $\mathcal{E}(.)$ is concave, or, equivalently, because $\mathcal{E}'(.)$ is decreasing.$\square$

9.8 Proof of Proposition 15.

If x_{n+1} were greater than x_n, $\overline{X}^{x_{n+1}}_{p_{n+1}}$ would exhibit FSD dominance over $\underline{X}^{x_n}_{p_n}$. Hence, x_{n+1} is lower than x_n and $\{x_n\}_{n\in\mathbb{N}}$ is a decreasing sequence. It is also bounded below by a. Consequently, it converges towards a limit $\ell \geq a$. Next, note that the two strongly indifferent simple lotteries $\underline{X}^{x_n}_{p_n}$ and $\overline{X}^{x_{n+1}}_{p_{n+1}}$have the same expected utility, i.e., we have:

$$p_n u(x_n) = p_{n+1} u\left(x_{n+1}\right) + \left(1 - p_{n+1}\right) \quad for \quad n = 0, 1, ... \tag{68}$$

and, consequently:

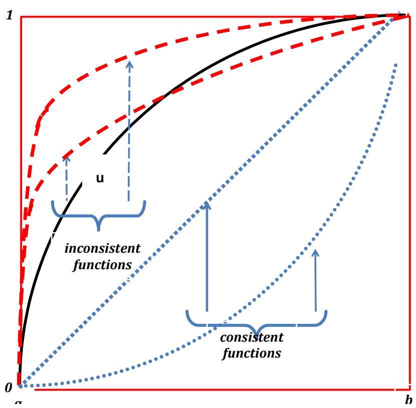
$$\pi u(w) = p_n u(x_n) + \textstyle\sum_{i=1}^{n}\left(1 - p_i\right) \quad for \quad n = 1, 2, ...$$

The above equality implies $S_n \overset{def}{=} \sum_{i=1}^{n}\left(1 - p_i\right) \leq \pi u(w)$. Since$\{S_n\}_{n\in\mathbb{N}^*}$ is an increasing sequence, it converges towards a limit $\Sigma \leq \pi u(w)$. As a consequence, $S_n - S_{n-1} = \left(1 - p_{n.}\right) \to 0$, i.e. $p_{n.} \to 1$. Moreover, since we have: $\underline{X}^{x_n+1}_{p_{n+1}} \prec \overline{X}^{x_{n+1}}_{p_{n+1}} \sim \underline{X}^{x_n}_{p_n}$, the sequence of binary lotteries $\{\underline{X}^{x_n}_{p_n}\}_{n\in\mathbb{N}}$ is decreasing and converges towards $\underline{X}^{\ell}_{1} = \delta(\ell)$. Similarly, $\{\overline{X}^{x_n}_{p_n}\}_{n\in\mathbb{N}^*}$ converges towards $\overline{X}^{\ell}_{1} = \delta\left(b - \ell\right)$.

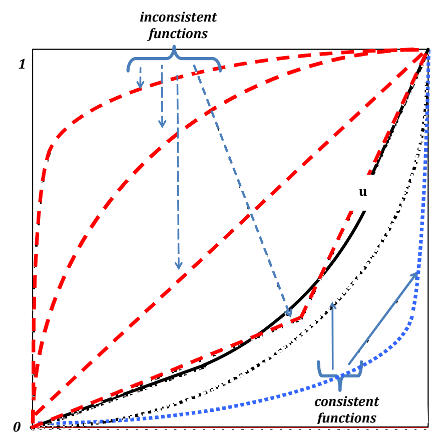
We now show that $\ell = a$. The proof is by contradiction. Indeed assume $\ell > a$. Then, since $\underline{X}^{x_n}_{p_n} \succ \delta\left(\ell\right)$, there exists a binary lottery $\underline{X}^{x_n^*}_{p_n}$ such that $\ell < x_n^* < x_n$, and $\underline{X}_{p_n}^{x_n^*} \sim \delta(\ell)$. Let x_{n+1}^* and p_{n+1}^* be defined by $\overline{X}_{p_{n+1}^*}^{x_{n+1}^*} \approx \underline{X}_{p_n}^{x_n^*}$. Since $\{\overline{X}_{p_n}^{x_n}\}_{n\in\mathbb{N}^*}$ converges towards $\delta\,(\ell)$, there exists an integer N, such that $m \geq N \Rightarrow \ell \leq x_m < x_{n+1}^*$ and $p_m \geq p_{n+1}^*$. This implies that $\overline{X}_{p_{n+1}^*}^{x_{n+1}^*}$ should be preferred to the $\overline{X}_{p_m}^{x_m}$s and, consequently, that $\delta(\ell)$ should be preferred to the $\overline{X}_{p_m}^{x_m}$s, what contradicts the fact that $\{\overline{X}_{p_n}^{x_n}\}_{n\in\mathbb{N}}$ is decreasing and converges towards $\delta(\ell)$. Hence $\ell = a$ and $\{S_n\}_{n\in\mathbb{N}}$ converges towards $\Sigma{=}\pi u(w)$. As a consequence, equality (31) is checked.

□

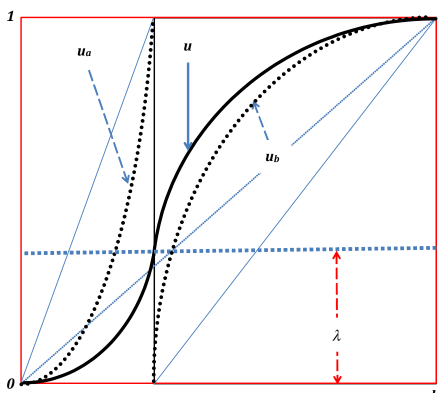
10 *Appendix 5. Figures*



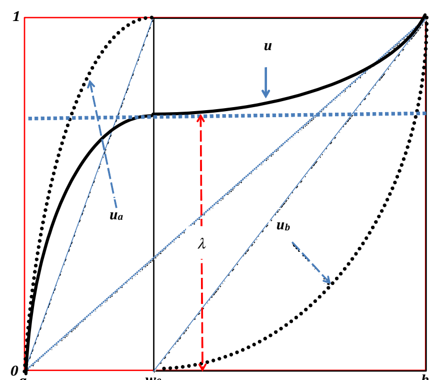
$Fig.1A$



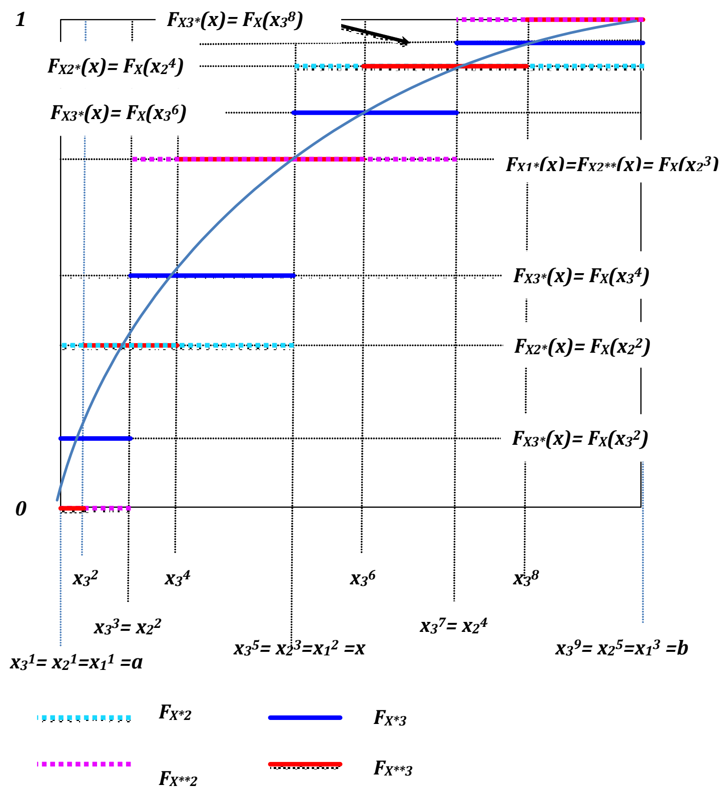
$Fig.1B$



$Fig.2\ A$



$Fig.2\ B$



1
$F_{X3^*}(x)= F_X(x_3^8)$
$F_{X2^*}(x)= F_X(x_2^4)$
$F_{X3^*}(x)= F_X(x_3^6)$
$F_{X1^*}(x)=F_{X2^{**}}(x)= F_X(x_2^3)$
$F_{X3^*}(x)= F_X(x_3^4)$
$F_{X2^*}(x)= F_X(x_2^2)$
$F_{X3^*}(x)= F_X(x_3^2)$
0
x_3^2
x_3^4
x_3^6
x_3^8
$x_3^3= x_2^2$
$x_3^7= x_2^4$
$x_3^1= x_2^1=x_1^1 =a$
$x_3^5= x_2^3=x_1^2 =x$
$x_3^9= x_2^5=x_1^3 =b$
F_{X^*2}
F_{X^*3}
$F_{X^{**}2}$
$F_{X^{**}3}$

Fig. 3